ALTERNATIVE DIRECT INTERPOLATION BOUNDARY ELEMENT METHOD APPLIED TO ADVECTIVE-DIFFUSIVE PROBLEMS WITH VARIABLE VELOCITY FIELD

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ABSTRACT

The wide range of physical phenomena of industrial interest which can be properly represented by advection-diffusion transport models motivates a constant effort in the development of new numerical methods capable of dealing with strong advective effects such as compressibility ones. The recent direct interpolation technique (DIBEM) proved to be an accurate and reliable tool for the representation of problems with constant velocity field and initial tests were also performed for problems with variable velocity field, where the results are reasonably satisfactory, but not so robust, since the integral relative to the velocity divergence, in general, seems to disturb the performance of the formulation. The current article presents a new formulation of the direct interpolation technique for solving variable velocity problems with non-zero velocity divergence. The accuracy of the new proposal is measured against a known analytical solution and, also, contrasted with the classical formulation of DIBEM and dual reciprocity technique (DRBEM) for the same case. Preliminary results show that the alternative DIBEM formulation proposed promotes a consistent improvement in precision, outperforming the two techniques in cross-comparison.

Keywords: diffusive-advective models; direct interpolation technique; boundary element method; variable velocity field

NOMENCLATURE

A auxiliar vector
b scalar product
c boundary shape coefficient
F radial basis function (RBF)
G^* Galerkin's Tensor
G BEM's traditional matrix
H BEM's traditional matrix
M advective transport matrix
n director cosine
N auxiliar variable
P auxiliar flux
q normal derivate of the scalar field
q^* normal derivate of the fundamental solution
r Euclidean distance
S auxiliar variable
u a generic scalar field
u^* Laplace/Poisson fundamental solution
v_i velocity field
X field point
Z auxiliar vector

Greek symbols

λ diffusion coefficient
α coefficient matrix
β approximation coefficient
Λ diagonal auxiliar matrix
ξ source point
ε average error
η DIBEM auxiliar variable of RBF approximation or DRBEM matrix
ψ DIBEM radial basis function primitive or DRBEM matrix
Ω domain representation
Γ boundary representation

Subscripts

i,j,k index notation
, differentiation operation

Superscripts

* variables associated to Green's problem
j discretization elements

INTRODUCTION

In the context of boundary element method (BEM) formulations based on radial basis approximations (Buhmann, 2003), the Dual Reciprocity Technique (DRBEM) proposed in the works of Nardini and Brebbia (1983) gives flexibility to the BEM, since it allows the use of a simpler Green's fundamental solution and provides an adequate approximation of the remaining domain integrals to the boundary using radial basis approximation. DRBEM was pioneered in its proposal and tested on several relevant scalar field problems (Partridge and Brebbia, 2012). In advective-diffusive models, in a general overview, the DRBEM technique has been barely tested in isolation, and its principal known limitation, when applied in pure form, consists in the restriction to low Péclet situations, in alignment with the work of Wrobel and DeFigueiredo (1991a). There is an alternative formulation, which circumvents this limitation, as can be seen in the work of Wrobel and DeFigueiredo (1991b). This formulation is based on decomposing the velocity field into an average plus fluctuation, where the average is treated via the fundamental correlated Green's solution and the fluctuation terms approximated by radial bases. In addition to the above, most work in the literature with the DRBEM technique on advection-diffusion models is simulated with constant element discretization.

As an alternative to DRBEM, Loeffler et al. (2015a) proposed the Direct Interpolation technique (DIBEM), also based on approximations by radial basis functions. The technique has already been extensively tested in relevant scalar field problems such as situations with domain actions (Loeffler et. al., 2015a), Helmholtz problems (Loeffler et al., 2015b), eigenvalue and self-equilibrium model in two dimensions (Loeffler & Mansur, 2017), and, also in the three-dimensional domain (Barbosa et al., 2019). The structure of the DIBEM proposal is, in a way, similar to the dual reciprocity, however, it proposes the approximation of the entire kernel of the domain integral, and mathematically, it resembles a classical interpolation procedure, which gives rise to its nomenclature. Regarding advective-diffusive models, systemic tests have recently been performed with constant velocity field situations (Pinheiro et al., 2022) in parallel with the dual reciprocity technique for moderate Péclet numbers. The DIBEM technique has been shown to be reasonably more accurate in this type of scenario and the direction of research pointing to variable velocity fields becomes natural.

In the context of advective-diffusive problems with variable velocity field some aspects are worth noting. First, it is interesting to point out that the variable velocity field, in the general case, demands the treatment of a non-null velocity divergent. In the case of DRBEM, since there is an approximation in the original structure of the domain integral of the advective side of the governing sentence (Partridge & Brebbia, 2012), this divergent remains implicit in the formulation. In contrast, in the classical DIBEM formulation, shown in Loeffler et al. (2020), the divergent appears in an explicit way and accounted for an exclusive integral, approximated by radial bases. Also, in Loeffler et al. (2020) preliminary tests of the classical DIBEM formulation are performed and contrasted with the DRBEM. Dual reciprocity demonstrates reasonably higher accuracy than DIBEM in these tests, which contradicts what was expected, since in constant velocity cases DIBEM had already demonstrated a solid advantage (Pinheiro et al., 2022). In this work DRBEM also demonstrates a sharper sensitivity to internal poles, comparable to that of DIBEM, a behavioral trait that is not present in the constant velocity tests. With the preliminary tests performed with the classical DIBEM formulation for variable velocity fields, it emerges in this paper the proposal of an alternative formulation for direct interpolation, with its absolute accuracy measured against the analytical solution and relative accuracy in parallel with the double reciprocity technique, which has proven robustness in problems of this nature with the use of linear elements.

DIBEM ALTERNATIVE FORMULATION

The governing equation of an advective-diffusive model in steady-state regime in an isotropic medium comes, physically, from the energy conservation sentence from continuum mechanics (Reddy, 2013) and can be algebraized in indicial notation, convenient in the form of Eq. (1) below.

$$\lambda u,_{ii}(X) = v_i(X)u,_i(X) \qquad (1)$$

In Eq. (1), λ represents the thermal diffusivity of the continuous medium, u, the scalar field of interest or primal variable and v_i the velocity field imposed on the domain. Strategies of the boundary element method that rely on approximations by radial basis functions make use of a simpler Green's solution, Poisson/Laplace’s for example. (Brebbia and Dominguez, 1994). Thus, to simplify the algebra, we will consider the properties of the unit medium (λ=1) and one can then write the strong integral formulation of the problem in the following way

$$\int_\Omega u,_{ii}(X)u^*(\xi,X)\,d\Omega = \int_\Omega v_i(X)u,_i(X)u^*(\xi,X)\,d\Omega \qquad (2)$$

Eq. (2) can be analyzed considering a diffusive (DS) side, the left of the equality, whose treatment is already known, and the right side, which accounts for advection effects (AS) which presents a challenging treatment. The formulations based on approximations via radial bases only differ in the treatment of the advective (AS) side. Thus, the diffusive side has its inverse integral formulation already well known, based on the best boundary element literature (Brebbia et al., 2012), and given by Eq. (3), where ξ represents the source points, X the coordinates of the field points, and q* and q the fluxes associated with the gradients of the potentials.

$$\mathrm{DS} = c(\xi)u(\xi) + \int_\Gamma u(X)q^*(\xi,X)d\Gamma - \int_\Gamma q(X)u^*(\xi,X)d\Gamma \quad (3)$$

Preliminary test results using the classical DIBEM formulation exhibited in Loeffler et al. (2020) and Loeffler et al. (2022) motivated the proposition of a formulation that could potentially respond better in cases of non-null divergent. The explicit nature of the divergent in the classical formulation appears to create numerical difficulties, and a proposal where there is an implicit accounting for this term, as the dual reciprocity does, could perhaps show accuracy improvement for some types of physical cases.

The demonstration of the new proposal begins by rewriting the advective side that must be treated by defining a variable b as the scalar product between the velocity field and the gradient of the primal field, as in Eq. (4).

$$\mathrm{AS} = \int_\Omega v_i(X)u_{,i}(X)u^*(\xi;X)d\Omega = \int_\Omega b(X)u^*(\xi;X)d\Omega \quad (4)$$

For the advective-diffusive model the definition of this scalar product, in a way, already refers to the dual reciprocity algorithm, which would approximate this term by radial basis functions. This fact gives rise to some structural similarities between the formulations of the classical DRBEM and the alternative DIBEM proposed here.

In the current proposal we will proceed by another path, however partially known. We will use the regularization procedure (Loeffler and Mansur, 2017), as in the version of the classical DIBEM, but this time, applied to the scalar product b(X). With this one can arrive at the following expression:

$$\int_\Omega b(X)u^*(\xi;X)d\Omega = \int_\Omega [b(X) - b(\xi)]u^*(\xi;X)d\Omega + \int_\Omega b(\xi)u^*(\xi;X)d\Omega \quad (5)$$

Eq. (5) rewrites the advective side in terms of a regularized domain integral, followed by an excess integral. The former will be approximated by radial basis by DIBEM, while the latter will be taken to the boundary using Galerkin's tensor. The steps of the treatment of each integral are explained in detail in the following.

Taking the entire kernel of the first domain integral of the right-hand side of Eq. (5), according to the direct interpolation technique, the approximation by radial bases is exposed in Eq. (6).

$$[b(X) - b(\xi)]u^*(\xi;X) = {}^{\xi}_{j}\alpha F^j(X^j;X) \quad (6)$$

Inserting the DIBEM approximation from Eq. (6) into the first integral on the right-hand side of Eq. (5), one then has, and naming a variable b'(X)= b(X)-b(ξ), one has:

$$\int_\Omega b'(X)u^*(\xi;X)d\Omega \cong \int_\Omega {}^{\xi}_{j}\alpha F^j(X^j;X)d\Omega \quad (7)$$

Now, by adopting a primitive ψ of the selected radial basis function such that $F^j = \psi,_{jj}$ and inserting this equality into the approximation in Eq. (7) it is possible to arrive at the following simplification. In Eq. (8) $(X^j;X)$ is the pair of interpolation points and the product $n_j\psi,_j$, can be condensed into a single variable called η^j.

$$\begin{aligned}\int_\Omega {}^{\xi}_{j}\alpha F^j d\Omega &= \int_\Omega {}^{\xi}_{j}\alpha\psi,_{jj}\, d\Omega \\ &= \int_\Omega \left({}^{\xi}_{j}\alpha\psi,_j\right)_{,j} d\Omega \\ &= {}^{\xi}_{j}\alpha \int_\Gamma n_j\psi,_j\, d\Gamma \\ &= {}^{\xi}_{j}\alpha \int_\Gamma \eta^j(X^j;X)d\Gamma\end{aligned} \quad (8)$$

Thus, the integrals of η^j can be easily calculated with numerical integration procedures, leaving as unknowns only the values of the α coefficients, which will be determined in due course.

The second integral of the right-hand side of Eq. (5) is conducted to the boundary using the concept of Galerkin's Tensor (G^*), which is the primitive of Green's solution (u^*). Thus adopting $G^*_{,kk} = u^*$, and substituting into the integral, one has:

$$\begin{aligned}b(\xi)\int_\Omega u^*(\xi;X)d\Omega &= b(\xi)\int_\Omega G^*{}_{,kk}d\Omega \\ &= b(\xi)\int_\Omega \left(G^*{}_{,k}\right)_{,k}d\Omega \\ &= b(\xi)\int_\Gamma n_k G^*{}_{,k}d\Gamma\end{aligned} \quad (9)$$

In Eq. (9), the directional derivative of the Galerkin's tensor $G^*(\xi;X)$ is given by:

$$G^*,_k(\xi;X)n_k(X) = {}^{\xi}P^X = \frac{1}{4\pi}\{0.5 - \ln[r(\xi;X)]\}r_k n_k \quad (10)$$

The discretization process applied to Eq. (9), considering the directional derivative in Eq. (10), leads to writing the vector (Z) as follows:

$$\begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} = \left[\begin{matrix} \left[\int_1^1 P_1 d\Gamma_1 + \cdots + \int_1^1 P_n d\Gamma_n\right] & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \left[\int_1^n P_1 d\Gamma_1 + \cdots + \int_1^n P_n d\Gamma_n\right] \end{matrix} \right] \begin{Bmatrix} u_1 \\ \vdots \\ u_n \end{Bmatrix} \tag{11}$$

Determination of the Alpha Matrix

Discretizing into elements and expanding the approximation of Eq. (6) to a discrete system with n degrees of freedom, and pre-multiplied by $[F]^{-1}$ results in Eq. (12).

$$^{\xi}\{\alpha\} = [F]^{-1}\ ^{\xi}[\Lambda]\{b'\} \tag{12}$$

The domain integral approximated by the DIBEM technique results in the discrete system formed by the product of two vectors $\{N\}$ and $\{\alpha\}$, which we adopt here to call A_ξ, as can be appreciated in the paper by Loeffler and Mansur, 2017. Since $\{N\}$ is the vector formed by the integrals of η^j, from Eq. (8), it is possible with the sentence of the coefficients α, Eq. (12), to write the approximate integral in the discrete domain as follows:

$$A_\xi = (N_1\ N_2\ \cdots\ N_n) \begin{pmatrix} {}_1^{\xi}\alpha \\ \\ {}_n^{\xi}\alpha \end{pmatrix} = (N_1\ N_2\ \cdots\ N_n) \begin{bmatrix} F^{11} & \cdots & F^{1n} \\ \vdots & \ddots & \vdots \\ F^{n1} & \cdots & F^{nn} \end{bmatrix}^{-1} \begin{bmatrix} {}_1^{\xi}\Lambda & & \\ & \ddots & \\ & & {}_n^{\xi}\Lambda \end{bmatrix} \{b'\} \tag{13}$$

One can group the entire product of terms, excluding {b'}, on the right-hand side in Eq. (13) into a vector $^{\xi}\{S\}$, as follows, from Eq. (14).

$$^{\xi}\{S\} = (S_{\xi 1}\ S_{\xi 2}\ \cdots\ S_{\xi n}) = (N_1\ N_2\ \cdots\ N_n) \begin{bmatrix} F^{11} & \cdots & F^{1n} \\ \vdots & \ddots & \vdots \\ F^{n1} & \cdots & F^{nn} \end{bmatrix}^{-1} \begin{bmatrix} {}_1^{\xi}\Lambda & & \\ & \ddots & \\ & & {}_n^{\xi}\Lambda \end{bmatrix} \tag{14}$$

The expansion of the discrete system with the scanning of the source points ξ, leads to the formation of the following matrix [S] that multiplies the vector {b'}, described by Eq. (15).

$$[S] = [N][F]^{-1}\ ^{\xi}[\Lambda] \tag{15}$$

At this point it is interesting to note that the integral approximated by direct interpolation becomes the product of the matrix [S], defined in Eq. (15) with the vector {b'}, which will be treated mathematically in due course. One can then write that the vector [A], in Eq. (16) that follows.

$$[A] = [S]\{b'\} \tag{16}$$

Determination of the Scalar Product

We now return to the scalar product b highlighted at the beginning of the proposed formulation. For the sake of simplicity, we highlight it here in Eq. (17).

$$b' = v_i u_{,i} \tag{17}$$

To better handle Eq. (17), we will resort to a process identical to that used in the dual reciprocity technique (DRBEM) to approximate the derivatives of the primal field u(X) (Partridge and Brebbia, 2012). To this end it is proposed that the field can be approximated, as in Eq. (18) by the product of a β coefficient by a radial basis function F selected. In this way the β coefficient can be determined by pre-multiplying by $[F^{-1}]$ as follows.

$$u = \beta F\ \therefore\ \beta = F^{-1}u \tag{18}$$

Now, in possession of the field approximation, one can then differentiate Eq. (18) with respect to the spatial coordinates, obtaining the directional derivatives $u_{,i}$ and using the expression of the coefficient β, to obtain a final expression for the derivatives as a function of the primal field u(X) and the chosen radial basis function as evidenced by Eq. (19).

$$u_{,i} = F_{,i}\beta\ \therefore\ \ u_{,i} = F_{,i}F^{-1}u \tag{19}$$

The expansion of Eq. (19) after discretization with n degrees of freedom results in a vector b, formed by scalar components, in the following form of Eq. (20):

$$\begin{Bmatrix} b'_1 \\ \vdots \\ b'_n \end{Bmatrix} = \left\{ \begin{bmatrix} v_1^1 & & \\ & \ddots & \\ & & v_1^n \end{bmatrix} \begin{bmatrix} F_{,1}^{11} & \cdots & F_{,1}^{1n} \\ \vdots & \ddots & \vdots \\ F_{,1}^{n1} & \cdots & F_{,1}^{nn} \end{bmatrix} \begin{bmatrix} F^{11} & \cdots & F^{1n} \\ \vdots & \ddots & \vdots \\ F^{n1} & \cdots & F^{nn} \end{bmatrix}^{-1} + \begin{bmatrix} v_2^1 & & \\ & \ddots & \\ & & v_2^n \end{bmatrix} \begin{bmatrix} F_{,2}^{11} & \cdots & F_{,2}^{1n} \\ \vdots & \ddots & \vdots \\ F_{,2}^{n1} & \cdots & F_{,2}^{nn} \end{bmatrix} \begin{bmatrix} F^{11} & \cdots & F^{1n} \\ \vdots & \ddots & \vdots \\ F^{n1} & \cdots & F^{nn} \end{bmatrix}^{-1} \right\} \begin{Bmatrix} u_1 \\ \vdots \\ u_n \end{Bmatrix} \tag{20}$$

For simplicity of mathematical writing, we will call $[M']$ the matrix that pre-multiplies the vector of unknowable field values, $\{u\}$, in Eq. (20). In this way, in a simpler way, one can say that the vector, $\{b'\}$, can be written as in Eq. (21):

$$\{b'\} = [M']\{u\} \tag{21}$$

With this the final matrix formulation in the new DIBEM formulation can be written as:

$$[H]\{u\} - [G]\{q\} = \{A + Z\} = [M]_{ID}\{u\} \tag{22}$$

The matrix $[M]_{ID}$ is here called the advective transport matrix, as it is connected here to the advective side of the governing equation of the phenomenon. The dual reciprocity technique arrives at a final discrete system of the same form, given below by Eq. (23). One focus of the current preliminary paper is to determine which of the transport matrices has a better constitution, and thus a better domain representation.

$$[\mathrm{H}]\{\mathrm{u}\} - [\mathrm{G}]\{\mathrm{q}\} = [\mathrm{H}\psi - \mathrm{G}\eta]\{\mathrm{b}\} = [\mathrm{M}]_{\mathrm{DR}}\{\mathrm{u}\} \quad (23)$$

METHODOLOGY

The numerical tests in the present paper are performed on perfectly structured meshes, illustrated by Figure 1, i.e., with the control variable $\alpha = 1$ in a square domain of unit dimensions, L=1. The radial basis approximations of all the exposed techniques are performed using the thin plate radial basis function, $r^2 \ln r$. The boundary discretization is done with linear elements for all formulations and the double nodes are spaced using $d=0.02l_e$.

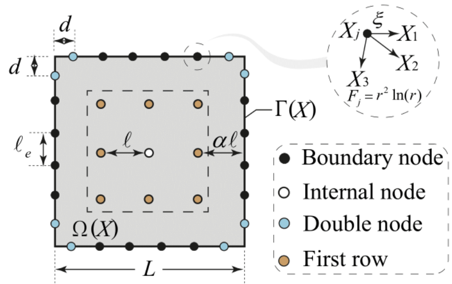

Figure 1- Computational Domain Scheme.

Preliminary tests are first performed with respect to convergence, where intermediate meshes are tested by gradually increasing the internal poles (IP) in the domain. As a sequence of the results a parametric analysis gradually accelerating the velocity field is performed to infer on the stability of the formulations in moderate advection physical scenarios. The results of the proposed alternative DIBEM formulation are compared simultaneously with the classical DIBEM formulation (Loeffler et al. 2020) and with DRBEM. The accuracy of the techniques is measured by the following mean error expression in Eq. (24), as the average of the three edges on which the fluxes are prescribed.

$$\varepsilon_{mean} = \frac{1}{n}\sum_{i=1}^{n}\frac{|u_i - u_{analytical}|}{\max u_{analytical}} \quad (24)$$

NUMERICAL TESTS

The numerical tests are generated on a compressible advective-diffusive case, with variable velocity field and nonzero velocity divergent. In order that the compressible problem can be adequately represented by a simple advection-diffusion equation such as Eq. (1), a coordinate transformation procedure shown in detail in the work of Loeffler and Dan (2004) was required. The boundary conditions and velocity field imposed on the domain are shown below in Figure 2.

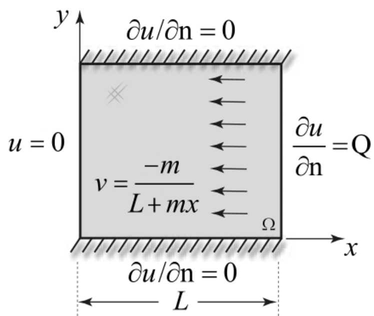

Figure 2- Sketch of the physical problem.

The accuracy of the numerical tests performed is measured in comparison with the analytical solution of the potential field, which can be found in the work of Dan et al. (2012). The parameter m present in the velocity field of Figure 2 and in the analytical solution serves to control the relative intensity of the advective effects.

The first numerical test performed consists of a convergence analysis where boundary meshes with 80, 120 and 160 boundary elements (BE) are fixed, and internal poles (IP) are gradually inserted to enrich the radial basis approximation of the tested formulations.

The graph in Figure 3(a) compares the results generated by the classical versus alternative DIBEM formulation. An interesting behavior of the new proposed formulation is observed, where the error levels decay rapidly to low levels even with a small number of interpolating poles. The graph in Figure 3(b) compares the alternative DIBEM formulation with the DRBEM. The error levels are similar for a more enriched domain of internal poles, however there is a relevant advantage with the use of few internal poles for the novel DIBEM formulation. The behavior demonstrated by the new proposal closely resembles that of DRBEM in advective-diffusive problems with constant velocity field. However, in the context of situations with variable velocity field the dual reciprocity apparently demands more internal poles for reasonable accuracy. Along these lines, this formulation demands tests in other physical situations, but it seems to demonstrate a lower demand for internal poles to achieve low error thresholds.

Next, a parametric analysis is performed with the parameter m, which controls the relative intensity of the advective effects in the governing model. This type of analysis is of great importance, because a central challenge for formulations that use approximations via radial bases is to maintain accuracy in the face of moderate and intense effects of the transport term. In this case, the graph in Figure 4(a) compares the classical DIBEM formulation with its alternative formulation, and Figure 4(b) compares this alternative formulation with the dual reciprocity technique.

First, in an overview of Figure 4(a) and 4(b) to compare the classical formulations of DIBEM with DRBEM, represented by black curves in both graphs, it can already be seen that in this case DRBEM takes a wide advantage over DIBEM in its classical version. Even with the greater sensitivity of DIBEM to internal pole arrangement and the possibility of improving the results by calibrating the parameter α, the dual reciprocity shows superiority in the case tested.

However, when evaluating the alternative formulation, it is consistently superior to the classical DIBEM as shown in Figure 4(a), and, also more accurate than the DRBEM as shown by the curves in Figure 4(b).

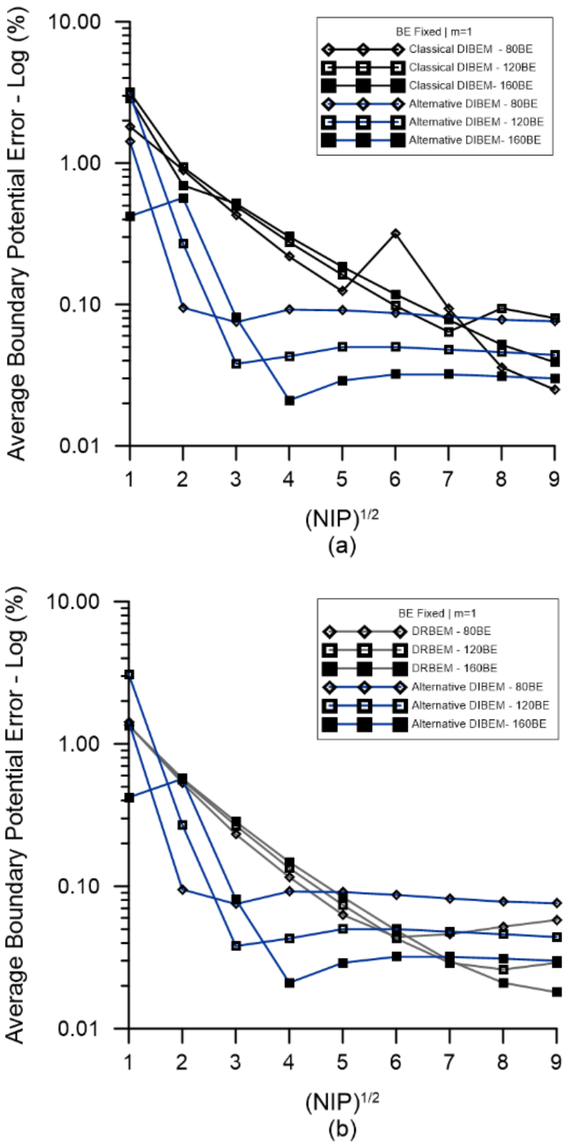

Figure 3- Convergence Analysis: (a) Classical DIBEM vs. Alternative DIBEM (b) Alternative DIBEM vs. DRBEM

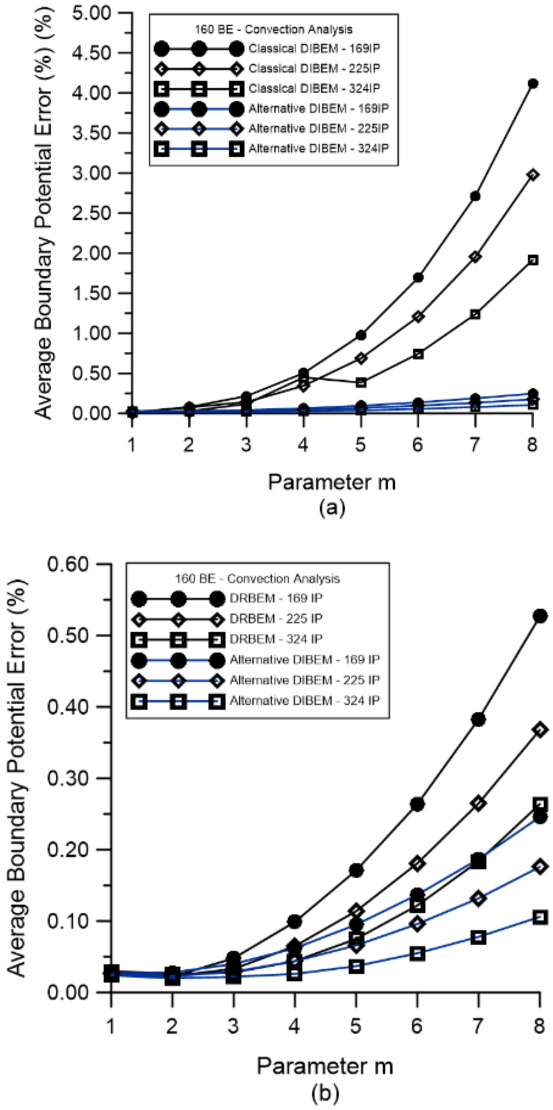

Figure 4- Parametric Analysis with m: (a) Classical DIBEM vs. Alternative DIBEM (b) Alternative DIBEM vs. DRBEM

CONCLUSIONS

The alternative direct interpolation formulation proposed in this paper proved to be accurate and robust in the preliminary tests performed, when compared to the classical direct interpolation techniques (DIBEM) and to the dual reciprocity (DRBEM), presenting results superior to both. The similarity between the final matrix equation of the dual reciprocity and the new formulation proposed here is remarkable, and it can be inferred that the final transport matrix generated by the proposed formulation tends to be more consistent, representing better the computational domain information than the one generated by DRBEM. The tests performed in this paper are initial and serve as motivation and generate a research vector in the direction of testing the new formulation in several other physical situations with variable velocity field to know its behavior systematically against strong advective effects and compressibility effects.

There is also an interest in extending future simulations to advective-diffusive models with the presence of reactive and source terms, to measure the flexibility and accuracy of this alternative formulation including other types of transport terms.

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