AN ANALYTICAL TEMPERATURE SOLUTION ANALYSIS FOR A MULTILAYER HEAT CONDUCTION PROBLEM

G. C. Oliveira,

A. P. Fernandes,

and G. Guimarães

[a]Universidade Federal de Uberlândia
Escola de Engenharia Mecânica
Bairro Santa Monica
CP. 34408, Uberlândia, Minas Gerais, Brasil
gabriela_costadeoliveira@hotmail.com

Received: October 29, 2014
Revised: November 30, 2014
Accepted: December 30, 2014

ABSTRACT

This paper presents a method of obtaining an analytic temperature solution for a two-layer heat conduction problem. Obtaining the temperature analytical solution for a multilayer heat conduction problem is not a direct method. The way to indentify the eigenvalues and to derive the Green function solution equation requires a different treatment since there are more than one domain to solve. This work presents a solution of a thermal two-layer problem based on Green's functions.

Keywords: analytical solution, multilayer, Green's functions, heat conduction

NOMENCLATURE

K_i	thermal conductivity in the i direction, i=1,2,3,…, W/(m.K)
L	length, m
b	layer length, m
q	heat flux, W/m^2
T	temperature, ºC
t	time, s
N	norm
G	Green function
x,y,z	cartesian coordinates, m

Greek symbols

α	thermal diffusivity, m^2/s
β	eigenvalue
λ	eigenvalue
η	eigenvalue
ν	fluid kinematic viscosity, m^2/s
ρ	density, kg/m^3

INTRODUCTION

It is proposed here to obtain the analytical solution for a two-layer thermal conduction problem using the Green's Functions (GF) method. An advantage in the use of integral solutions by GF is the possibility to build multidimensional solutions from 1D cases without additional difficulties. It can be observed, in this sense, that 2D and 3D solutions are completely equivalent to 1D equation, and therefore they can be obtained from product solutions in different directions 1D (Fernandes, 2009).

Analytical solutions are an important tool for solution of engineering problems, since they can be used: to validate approximate solutions; to facilitate the analysis and understanding of physical problems; in construction of physical problems; in construction of new numerical algorithms such as the transient method of surface element or in direct application in real problems reducing the computational cost and exact solutions of the model studied. (Fernandes, 2009).

THEORY

One-dimensional Transient Thermal Problem X22

Initially, it is presented the 1D transient problem X22 that is used as an auxiliar problem to solve the multilayer problem. This problem is shown on Fig. 1.

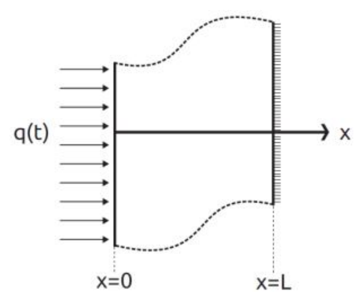

Figure 1. Single layer plate subjected to the heat flux, q(t) at x = 0, while the opposite surface, x = L, is isolated.

The problem represented by Fig. 1 is given by:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1a}$$

subject to the boundary conditions:

$$\left.\frac{\partial T}{\partial x}\right|_{x=L} = 0 \quad -k\left.\frac{\partial T}{\partial x}\right|_{x=0} = q(t); \tag{1b}$$

and to the initial condition

$$T(x,0) = F(x) = T_0 \tag{1c}$$

The solution of Eqs. (1a)-(1c) can be obtained by using Green's functions, it means

$$T(x,t) = T_0 + \alpha \int_0^t G(x,t\,|\,0,\tau)\frac{q(t)}{k}d\tau \tag{2}$$

Where $G(x,t\,|\,0,\tau)$ is the Green function that is given by (Fernandes, 2009)

$$G_{X22}(x,t\,|\,x',\tau) = \frac{1}{L}\left[1 + 2\sum_{m=1}^{\infty} e^{-\left(\frac{m\pi}{L}\right)2\alpha(t-\tau)} x\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{m\pi x'}{L}\right)\right] \tag{3}$$

It ca be observed that the problem described by the Eqs. (1) is equivalent to the problem with a volumetric heat source applied at $x=0$ that is $g(x,t) = q(t)\delta(x-0)$ with zero heat flux at $x=0$ (Hahn and Ozisik, 2012). Therefore, the problem given by Eqs. (1) could be described by the following equations:

$$\frac{\partial^2 T}{\partial x^2} + g(x,t) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{4a}$$

Subject to the boundary conditions

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0; \quad \left.\frac{\partial T}{\partial x}\right|_{x=L} = 0 \tag{4b}$$

Using the Green function method, the solution of (4) is given by:

$$T(x,t) = T_0 + \alpha\int_0^t\int_0^L G(x,t|x',\tau) \times \frac{q(t)\delta(x'-0)}{k}dx'd\tau \tag{5}$$

One-dimensional Transient Thermal Problem X2C12

The problem of heat conduction 1D X2C12 shown in Fig. 2 describes a two layer plate accounting for the heat flux by using $g(x,t) = q(t)\delta(x-0)$ with isolate condition at the ends and can be written as.

The problem represented by Fig. 2 is given by:

$$\frac{\partial^2 T_1}{\partial x^2} + g(x,t) = \frac{1}{\alpha}\frac{\partial T_1}{\partial t} \tag{6a}$$

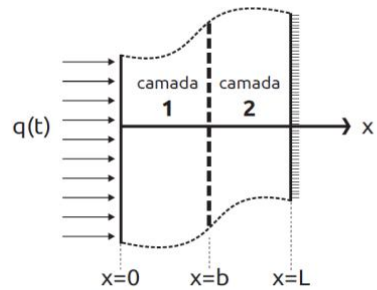

Figure 2. Two-layer plate subjected to the heat flux, q(t) at x=0, while the opposite surface, x = L, is isolated.

Subject to the boundary conditions:

$$-k_1\left.\frac{\partial T_1}{\partial x}\right|_{x=0} = 0; \quad -k_2\left.\frac{\partial T_2}{\partial x}\right|_{x=L} = 0; \tag{6b}$$

and to the continuity conditions

$$T_1|_{x=b} = T_2|_{x=b}; \quad -k_1\left.\frac{\partial T_1}{\partial x}\right|_{x=b} = -k_2\left.\frac{\partial T_2}{\partial x}\right|_{x=b} \tag{6b}$$

and to the initial condition

$$T_1(x,0) = T_2(x,0) = F(x) = T_0 \tag{6d}$$

The solution in each region *i* is then written as

$$T_i(x,t) = \sum_{j=1}^{M}\left\{\int_{x_j}^{x_{j+1}} G_{ij}(x,t\,|\,x',0)F_j(x')dx' + \alpha_j\int_0^t\int_{x_j}^{x_{j+1}} G_{ij}(x,t\,|\,x',\tau)\frac{g_j(x',\tau)}{k_j}dx'd\tau\right\} \tag{7}$$

Where $x_j \le x \le x_{j+1}$; to $j = 1,2,3,\ldots,M$ are the limits of each layer, and $G_{ij}(x,t\,|\,x',\tau)$ is the Green's function for the two-layer body.

If M = 1, the single layer solution can be given by Eq. (8) that is algebraically equal to Eq. (5). It means

$$T_1(x,t)=\alpha_1\int_0^t\int_{x_1}^{x_2}G_{11}(x,t\,|\,x',\tau)\frac{g_{11}(x',\tau)}{k_{(1)}}dx'd\tau \qquad (8)$$

If $M=2$, $0\le x\le b$ and $b\le x\le L;$ the solutions for T_1 e T_2 given by Eqs. (9) and (10) respectively are given by

$$T_1(x,t)=\alpha_1\int_0^t\int_{x_1}^{x_2}G_{(11)}(x,t\,|\,x',\tau)\frac{g_1(x',\tau)}{k_1}dx'd\tau + \alpha_2\int_0^t\int_{x_2}^{x_3}G_{12}(x,t\,|\,x',\tau)\frac{g_2(x',\tau)}{k_2}dx'd\tau \qquad (9)$$

and

$$T_2(x,t)=\alpha_1\int_0^t\int_{x_1}^{x_2}G_{(21)}(x,t\,|\,x',\tau)\frac{g_1(x',\tau)}{k_1}dx'd\tau + \alpha_2\int_0^t\int_{x_2}^{x_3}G_{22}(x,t\,|\,x',\tau)\frac{g_2(x',\tau)}{k_2}dx'd\tau \qquad (10)$$

It can be observed that $g(x,t)=q(t)\delta(x-0)$, therefore the second term of the Eq.(9) and Eq.(10) are null.

The Green function Gij is given by Haji-Sheikh and Beck (2002)

$$G_{ij}(x,t\,|\,x',\tau)=\sum_{n=1}^{\infty}e^{-\lambda_n^2(t-\tau)}\frac{1}{N_x}X_{in}(x)X_{jn}(x'), \qquad (11)$$

where X_{in} and X_{jn} are the eigenfunctions and λ_n are the eigenvalues.

The N_x norm is defined by

$$N_x=\sum_{j=1}^{M}\int_{x_j}^{x_{j+1}}\left[X_{jn}(x')\right]^2dx' \qquad (12)$$

Therefore, the temperature solution in the range $\left[x_1;x_2\right]$ has the following form:

$$T_1(x,t)=\frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{X_{1n}}{N_x}\int_0^t e^{-\lambda_n^2(t-\tau)}\int_{x_1}^{x_2}X_{1n}(x')\times q(t)\delta(x'-0)dx'd\tau \qquad (13)$$

$$T_1(x,t)=\frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{X_{1n}(x)X_{1n}(0)}{N_x}\int_0^t q(t)e^{-\lambda_n^2(t-\tau)}d\tau$$

and in the range $\left[x_2;x_3\right]$:

$$T_2(x,t)=\frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{X_{2n}}{N_x}\int_0^t e^{-\lambda_n^2(t-\tau)}\int_{x_1}^{x_2}X_{1n}(x')\times q(t)\delta(x'-0)dx'd\tau \qquad (14)$$

$$T_2(x,t)=\frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{X_{2n}(x)X_{1n}(0)}{N_x}\int_0^t q(t)e^{-\lambda_n^2(t-\tau)}d\tau$$

The eigenfunctions $X_1=X_{1n}(x)$ and $X_2=X_{2n}(x)$ and respective eigenvalues are obtained using separation of variables method. In this sense, assuming a separation of $T(x,t)$ in two spatially and time dependent functions of a single variable each in the form

$$T_1(x,t)=X_1(x)\mathrm{T}_1(t) \qquad (15a)$$

$$T_2(x,t)=X_2(x)\mathrm{T}_2(t) \qquad (15b)$$

Substituing Eq. (15a) in Eqs. (6a) and Eq. (15a) in Eqs. (6b) yelds

$$\frac{\partial^2X_1}{\partial x^2}+\gamma^2X_1=0; \qquad (16a)$$

$$\frac{\partial^2X_2}{\partial x^2}+\eta^2X_2=0 \qquad (16b)$$

where

$$\gamma^2=\frac{\lambda^2}{\alpha_1};\eta^2=\frac{\lambda^2}{\alpha_2}$$

And the solutions for X_1 and X_2 are

$$X_1=A\cos(\gamma x)+Bsen(\gamma x); \qquad (17a)$$

$$X_2=C\cos(\eta x)+Dsen(\eta x); \qquad (17b)$$

It can be observed that Eq. (17a) must satisfy the boundary condition at $x=0$

$$-k_1\left.\frac{\partial T_1}{\partial x}\right|_{x=0}=0; \qquad (18)$$

After substituting the eigenfunction X_1 Eq (17a) in Eq (18) and solving the expression, we obtain $B=0$ and without loss of generality it is concluded that $A=1$ (Özi¸sik, 1993). Soon,

$$X_1 = \cos(\gamma x) \tag{19}$$

Then the boundary conditions at $x = b$ must be satisfied , it means

$$X_1 |_{x=b} = X_2 |_{x=b}; \tag{20a}$$

$$-k_1 \left.\frac{\partial X_1}{\partial x}\right|_{x=b} = -k_2 \left.\frac{\partial X_2}{\partial x}\right|_{x=b} \tag{20b}$$

After substituting X_1 and X_2 in equation (20a)-(20b) yelds

$$\cos(\gamma b) - C\cos(\eta b) - Dsen(\eta b) = 0 \tag{21}$$

and

$$-\left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right) sen(\gamma b) + Csen(\eta b) - D\cos(\eta b) = 0 \tag{22}$$

As the boundary condition at $x = L$ is

$$-k_2 \left.\frac{\partial X_2}{\partial x}\right|_{x=L} = 0; \tag{23}$$

after substituing X_2 in Eq. (23) , Eqs., (21), (22) and (23) , can be written in matrix form as

$$\begin{pmatrix} \cos(\gamma b) & \cos(\eta b) & sen(\eta b) \\ -Ksen(\gamma b) & sen(\eta b) & -\cos(\eta b) \\ 0 & -\eta sen(\eta L) & \eta\cos(\eta L) \end{pmatrix} \begin{bmatrix} 1 \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

where

$$K = \left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right)$$

Coefficients C and D are then determined by solving the linear system given by equation (24). It means

$$C = \cos(\eta b)\cos(\gamma b) + \left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right) sen(\gamma b) sen(\eta b)$$

$$D = \cos(\gamma b) sen(\eta b) - \left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right) sen(\gamma b)\cos(\eta b)$$

Since all coefficient have been calculated the eigenfunctions coefficients X_1 and X_2 are given by

$$X_1 = \cos(\gamma x) \tag{25}$$

$$X_2 = \left[\cos(\eta b)\cos(\gamma b) + \left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right) sen(\gamma b) sen(\eta b)\right]\cos(\eta x)$$

$$+\left[\cos(\gamma b) sen(\eta b) - \left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right) sen(\gamma b)\cos(\eta b)\right] sen(\eta x) \tag{26}$$

And the temperature solution can be given by

$$T_1(x,t) = \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{X_{1n}}{N_x}\int_0^t e^{-\lambda_n(t-\tau)}\int_{x_1}^{x_2} X_{1n}(x')$$

$$\times q(t)\delta(x'-0)dx'd\tau$$

$$T_1(x,t) = \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{X_{1n}(x)X_{1n}(0)}{N_x}$$

$$\times\int_0^t q(t)e^{-\lambda_n(t-\tau)}d\tau$$

$$T_1(x,t) = \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{\cos(\gamma x)\cos(0)}{N_x}$$

$$\times\int_0^t q(t)e^{-\lambda_n(t-\tau)}d\tau \tag{27a}$$

and

$$T_2(x,t) = \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{X_{2n}}{N_x}\int_0^t e^{-\lambda_n(t-\tau)}\int_{x_1}^{x_2} X_{1n}(x')$$

$$\times q(t)\delta(x'-0)dx'd\tau$$

$$T_2(x,t) = \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{X_{2n}(x)X_{1n}(0)}{N_x}$$

$$\times\int_0^t q(t)e^{-\lambda_n(t-\tau)}d\tau$$

$$T_2(x,t) = \frac{\alpha_1}{k_1}\sum_{n=1}^{\infty}\frac{1}{N_x}\left\{\left[\cos(\eta b)\cos(\gamma b) + \left(\frac{k_1}{k_2}\right)\right.\right.$$

$$\left.\times\left(\frac{\gamma}{\eta}\right) sen(\gamma b) sen(\eta b)\right]\cos(\eta x)$$

$$+\left[\cos(\gamma b)sen(\eta b)-\left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right)sen(\gamma b)\cos(\eta b)\right.$$

$$\left.\times sen(\eta x)\right\}\cos(0)\int_0^t q(t)e^{-\lambda_n(t-\tau)}d\tau \quad (27b)$$

where the norm, N_x is given by:

$$N_x=\int_{x_1}^{x_2}\left[X_{1n}(x')\right]^2dx'+\int_{x_2}^{x_3}\left[X_{2n}(x')\right]^2dx'$$

$$N_x=\int_0^b\left[\cos(\gamma x')\right]^2dx'+\int_b^L\left\{\left[\cos(\eta b)\cos(\gamma b)+\left(\frac{k_1}{k_2}\right)\right.\right.$$

$$\left.\times\left(\frac{\gamma}{\eta}\right)sen(\gamma b)sen(\eta b)\right]\cos(\eta x')$$

$$+\left[\cos(\gamma b)sen(\eta b)+\left(\frac{k_1}{k_2}\right)\left(\frac{\gamma}{\eta}\right)\right.$$

$$\left.\left.\times sen(\gamma b)\cos(\eta b)\right]sen(\eta x')\right\}^2dx' \quad (28)$$

The equation for determination of the eigenvalues is obtained from the requirement that in Eq. (24) the determinant of the coefficients should vanish. Then, the eigenvalues are roots of the following transcendental equation:

$$\begin{vmatrix}\cos(\gamma b) & \cos(\eta b) & sen(\eta b)\\ -Ksen(\gamma b) & sen(\eta b) & -\cos(\eta b)\\ 0 & -\eta sen(\eta L) & \eta\cos(\eta L)\end{vmatrix}=0$$

and

$$\tan(\gamma b)=-K\tan[\eta(b-L)] \quad (29)$$

Equation (29) can be solved by using various mathematical methods. (Beck, 1992) and (Haji-Sheik and Beck, 2000) present solutions to the transcendental equation based on asymptotic approximations. For each eigenvalue, there are welldefined upper and lower values for η within which only one eigenvalue is located. These limits are the ordered locations of the asymptotes for the right side and for the left side of each equation. The asymptotes will be located where the denominators of the right side and of the left side become zero. Figure (3), shows the behavior of these asymptotes.

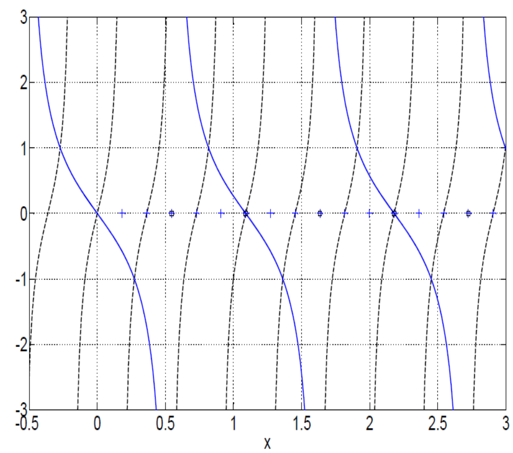

Figure 3. Representation of the asymptotes curves for η.

RESULTS AND DISCUSSION

The solution of Eq. (27) is shown in Fig. 4 considering the heat flux prescribed equal $4\times10^5\left[\frac{W}{m^2}\right]$, $T_0=0\left[^oC\right]$, $L=5\times10^{-2}[m]$, $b=\frac{L}{2}[m]$, $\alpha_1=18{,}8\times10^{-6}$, $\alpha_2=117\times10^{-6}$, $K_1=64\left[\frac{W}{mk}\right]$ and $K_2=401\left[\frac{W}{mk}\right]$.

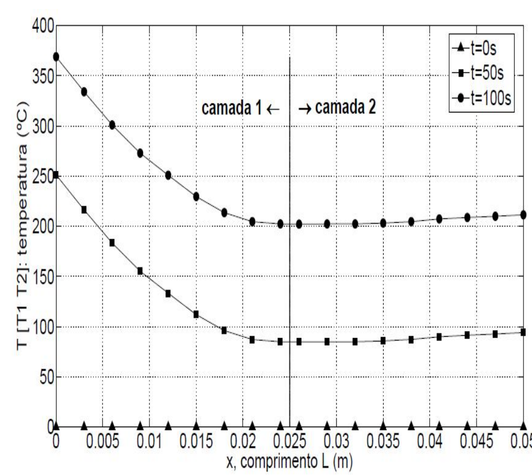

Figure 4. Spatial temperature for three different times. Two-layer body.

The two-layer solution can be verified by considering the one layer solution (Fig. 5). It means if both layers have the same thermal properties the thermal problem can be described as a one single layer problem. For comparison, a single layer solution is presented (Fig. 5). These results are then compared with the solution of two-layer case, but considering he same thermal properties (Fig. 6).

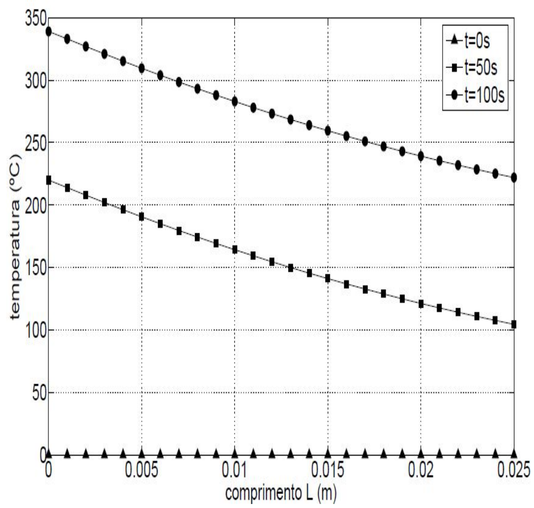

Figure 5. Spatial temperature for three different times. Single layer body.

In both cases the same parameter are used. It means, $4\times10^5\left[\frac{W}{m^2}\right]$, $T_0=0\left[{}^oC\right]$, $L=5\times10^{-2}\left[m\right]$, $b=\frac{L}{2}\left[m\right]$, $\alpha_1=18,8\times10^{-6}$, $\alpha_2=117\times10^{-6}$, $K_1=64\left[\frac{W}{mk}\right]$ and $K_2=401\left[\frac{W}{mk}\right]$. The absolute deviate between the solutions of the problem $X22$ (Fig. 5) and the problem $X2C12$ (Fig. 6) is shown in (Fig. 7). It can be observed a maximum difference of $0,14({}^oC)$ with a percentage error of $0,04\%$.

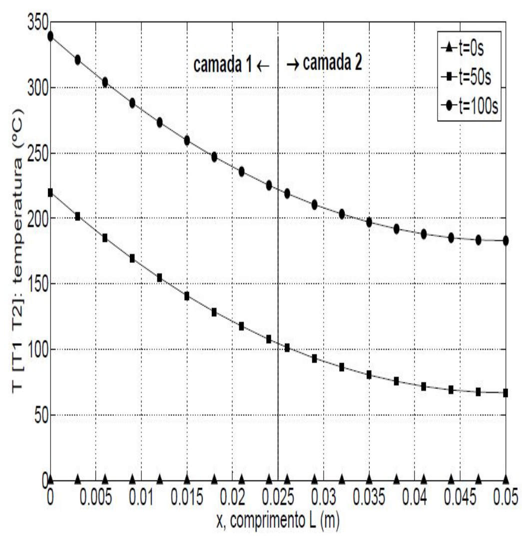

Figure 6. Spatial temperature for three different times. Two-layer body with the same thermal properties.

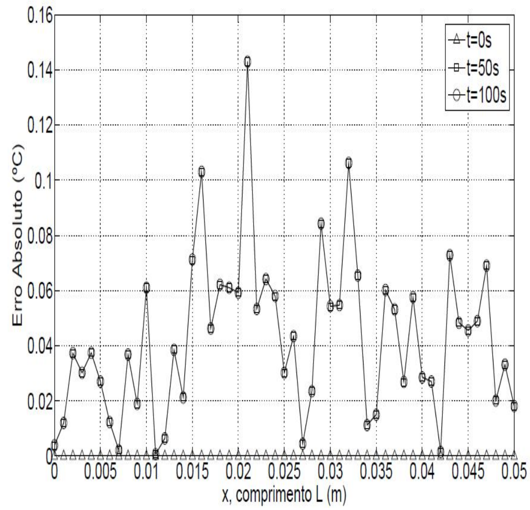

Figure 7. Absolute error between the problems X22 and X2C12.

CONCLUSIONS

A temperature solution for a two-layer body has been presented. The analytical solution has been verified by using one layer case. Calculating the spatial eigenvalues is the main difficulty that one must consider when dealing with problems of this type. The procedure presented here can be extended for multilayer bodies.

ACKNOWLEDGEMENTS

The authors would like to thank the agencies CAPES, FAPEMING and CNPq.

REFERENCES

Beck, J., 1992, *Heat Conduction using Green's Functions in Computational and Physical Processes in Mechanics and Thermal Sciences*, Hemisphere Publishing Corp.

Cole, K., Haji-Sheikh, A., Beck, J. V., and Litkouhi, B., 2010, *Heat Conduction using Green's Function. Series in Computational and Physical Processes in Mechanics and Thermal Sciences*, Taylor & Francis.

Fernandes, A. P., 2009, Funções de Green: Soluções Analíticas Aplicadas a Problemas Inversos em Condução de Calor, Master Thesis, Universidade Federal de Uberlândia, Minas Gerais. *(in Portuguese)*

Hahn, D., and Ozisik, N., 2012, *Heat Conduction*, Wiley.

Haji-Sheik, A., and Beck, J. V., 2000, An Efficient Method of Computing Eigenvalues in Heat Conduction, International Journal of Computation and Methodology, Vol. 38, No. 2, pp. 133-156.

Haji-Sheikh, A., 2014, *Two-layer Slab with Perfect Contact Between Layers; with Zero in Heat*

Flux at one Boundary, Zero Heat Flux at Other Boundary, ExACT.

Haji-Sheikh, A., and Beck, J., 2002, Temperature Solution in Multi-dimensional Multi-layer Bodies, International Journal of Heat and Mass Transfer, Vol. 45, No. 9, pp. 1865- 1877.

Özisik, M. N., 1993, *Heat Conduction, 2nd Edition*, Wiley-Interscience.