ABOUT THE EXACT SOLUTION IN TWO PHASE-STEFAN PROBLEM

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ABSTRACT

Two cases of the two - phase Stefan problem in a semi - infinite slab are presented here: one has heat flux boundary condition proportional to $t^{-½}$ and the other has constant temperature boundary condition. In these two cases the exact solution exists, the relationship between the two boundary conditions is presented here, and the equivalence between the two problems is shown.

Keywords: two phase Stefan Problem, exact solution, heat flux boundary condition, temperature boundary condition.

NOMENCLATURE

c	specific heat, J/kg ºC
k	thermal conductivity, W/m ºC
L	latent heat of fusion, J/kg
$q(t)$	heat flux, W/m^2
$s(t)$	interface position, m
t	time independent variable, s
$T(x,t)$	temperature profile of the phase change material, ºC
T_f	melting temperature, ºC
T_0	initial temperature, ºC
$u(x,t)$	difference between $T(x,t)$ and T_f, ºC
u_0	difference between T_0 and T_f, ºC
x	spatial independent variable, m
Ste	Stefan number

Greek symbols

α	thermal diffusivity, m^2/s
ρ	density, kg/m^3
ξ	similarity variable, m/s$^{1/2}$
λ	parameter to be determined, dimensionless

Subscripts or superscripts

L	liquid phase
S	solid phase
f	melting point
0	initial state

INTRODUCTION

The free boundary problem presents many applications in physics and engineering. In particular, the so called “One and Two Phase Stefan Problem.”(Alexiades and Solomon, 1993) in a semi – infinite slab is of great interest.

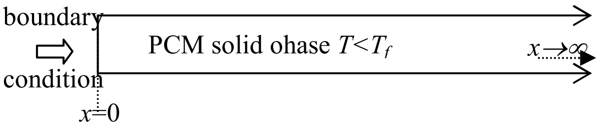
The one - phase Stefan Problem takes place when the initial phase is solid (liquid) at the melting point, the material is heated (cooled) from the side and, as a consequence, the temperature increases (decreases). The substance change to liquid (solid) phase in the vicinity of the boundary, leaving the rest of the solid (liquid) at the same initial temperature. This case was studied in a previous work (Boucíguez *et al*, 2006)

The two - phase Stefan Problem takes place when the initial phase is solid (liquid) at a temperature less (greater) than the melting one, the material is heated (cooled) from the side and as a consequence the temperature increases (decreases) to reach the melting point and then the liquid (solid) appears. In this case, the two phases: liquid and solid present a temperature distribution. These two temperature distributions and the interface position (free boundary) are unknown.

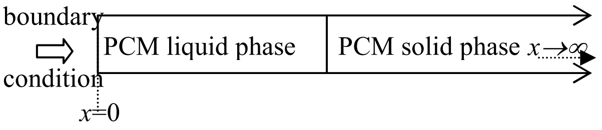
These two cases have exact solution when the condition in the boundary is: a constant temperature or the heat flux proportional to $t^{-½}$.

In a previous work (Boucíguez et al, 2006) it was shown that in the one - phase Stefan problem, the two boundary conditions are equivalent. Now, the equivalence between them will be shown for the two - phase Stefan problem.

Figure 1 shows a scheme of the phase change material when solid is the initial condition, (a) at time t=0, and (b) at time t>0. If the initial condition is liquid the situation is equivalent, the words solid and liquid and the inequalities must be replaced in the figure.



(a) phase change material at t=0



(b) phase change material at t>0

Figure 1. Scheme of the phase change material.

In a previous work (Lozano et al, 2003) it was observed that when the flux is $q(t) = q_0/t^{1/2}$, the temperature in the fixed face stays constant. In fact, in figures 2 to 5, the temperature distribution for four different times (5, 10, 15, and 20 hours) are shown when q_0 is equal to 100, 500, 1000, and 2000 Ws^{½}/m^2, respectively. The corresponding temperatures in the fixed face (x=0) are constant and equal to 0.0001, 0.0024, 0.0098, and 0.0392 ºC respectively. These figures suggest a connection between the two boundary conditions and their equivalence has been shown here; that is to say: one is a consequence of the other and reciprocally

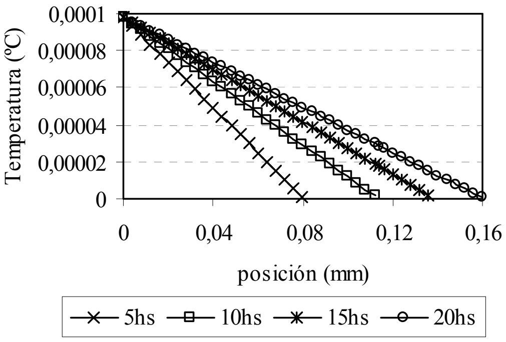


Figure 2. Temperature distribution for q_0=100 Ws^{½}/m^2.

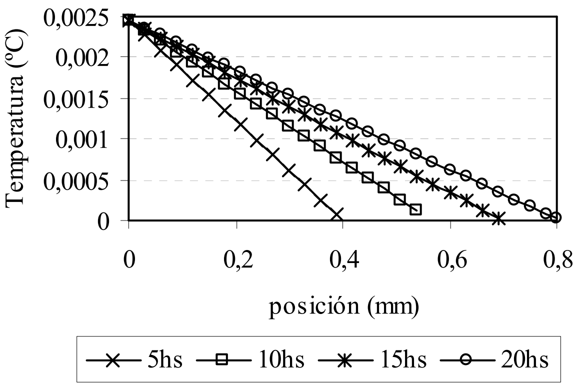


Figure 3. Temperature distribution for q_0=500 Ws^{½}/m^2.

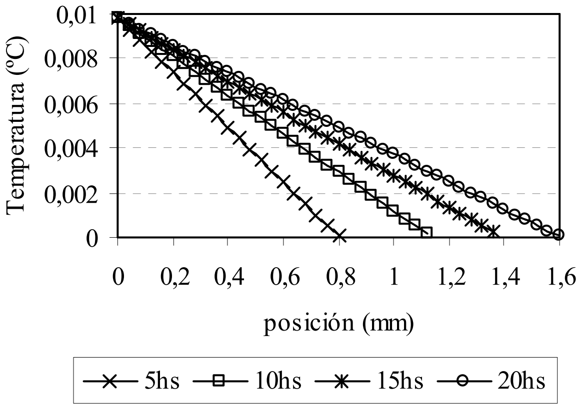


Figure 4. Temperature distribution for q_0=1000 Ws^{½}/m^2.

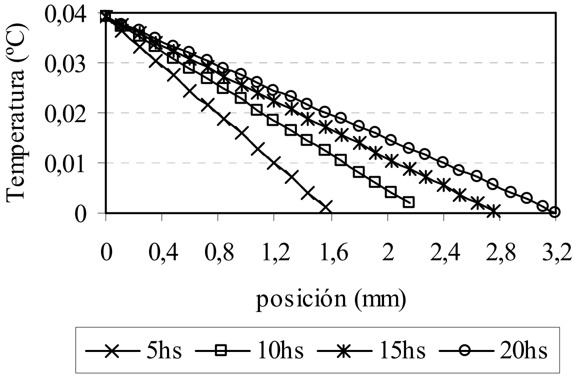


Figure 5. Temperature distribution for q_0=2000 Ws^{½}/m^2.

MATHEMATICAL FORMULATION

The two phase Stefan problem can be formulated as a heat conduction problem in a semi – infinite slab, with two kind of boundary condition: constant temperature or heat flux proportional to $t^{-½}$.

The problem is completely described by the following equation:

$$\frac{\partial u^L}{\partial t} = \alpha_L \frac{\partial^2 u^L}{\partial x^2}, \quad 0 < t < \tau, \ 0 < x < s(t) \tag{1.a}$$

$$\frac{\partial u^S}{\partial t} = \alpha_S \frac{\partial^2 u^S}{\partial x^2}, \ 0 < t < \tau, \quad x > s(t) \tag{1.b}$$

$$u^L(s(t),t) = u^S(s(t),t) = 0, \ \forall t > 0, \ \ 0 < t < \tau \tag{2}$$

$$\rho\, L \frac{ds(t)}{dt} = -k_L \frac{\partial u^L(s(t)^-,t)}{\partial x} + k_S \frac{\partial u^S(s(t)^+,t)}{\partial x}, \quad \forall t > 0, \quad 0 < t < \tau \tag{3}$$

$$u^S(x,0) = u_S < 0, \ \ 0 < x \le \infty \tag{4}$$

$$s(0) = 0 \tag{5}$$

and for the constant temperature case

$$u^L(0,t) = u_L > 0, \quad \forall t > 0, \quad 0 < t < \tau \tag{6.1}$$

or for the heat flux proportional to $t^{-½}$

$$k_L \frac{\partial u^L(0,t)}{\partial x} = -\frac{q_0}{\sqrt{t}}, \ \ \forall t > 0, \ 0 < t < \tau, \ q_0 > 0 \tag{6.2}$$

Where α is the thermal diffusivity, *k* is the thermal conductivity, *ρ* is the density, *c* is the specific heat and *L* is the latent heat of fusion. The index *L* and *S* (upper or sub) denote liquid and solid phase respectively.

The function *s(t),* unknown a priori, is the interface position as a function of *t*; and $u(x,t)=T(x,t)-T_f$, is the difference between the substance temperature *T(x,t)* and the fusion temperature T_f. At the same way $u_L=T_L-T_f$, is the difference between the temperature at the fixed face T_L and T_f.

The Eq. 3 is the Stefan's equation, it represent the energy conservation on the interface position.

The Eqs. (6.1) and (6.2), show the two possible boundary conditions: constant temperature (6.1) and heat flux ~ $1/t^{½}$ (6.2). They are the only two cases where this problem has exact solution.

In the two cases, the analytical solution is obtained introducing the similarity variable ξ, (Alexiades and Solomon, 1993), defined by:

$$\xi = \frac{x}{\sqrt{t}} \tag{7}$$

The exact solution of the problem is obtained replacing (7) into Eqs. (1) to (6), hence the interface position results:

$$s(t) = 2\lambda\sqrt{\alpha_L t} \tag{8}$$

Where λ is a parameter to be determined for each case. For the constant temperature boundary condition, λ is given by the equation:

$$\lambda \cdot \sqrt{\pi} = \frac{Ste^L}{\exp(\lambda^2)\cdot erf(\lambda)} - \frac{Ste^S}{\upsilon \exp(\upsilon^2\lambda^2)\cdot erf(\upsilon\lambda)} \tag{9.1}$$

and for the flux boundary condition, λ is given by the equation:

$$\lambda \cdot \sqrt{\pi} = \frac{q_0}{\rho \cdot L}\sqrt{\frac{\pi}{\alpha_L}}\,\frac{1}{\exp(\lambda^2)} - \frac{Ste^S}{\upsilon \exp(\upsilon^2\lambda^2)\cdot erf(\upsilon\lambda)} \tag{9.2}$$

where $Ste^L = \frac{c_L}{L}u^L$, $Ste^S = \frac{c_S}{L}u^S$ are the Stefan's number (liquid and solid, respectively) and $\upsilon = \sqrt{\frac{\alpha_L}{\alpha_S}}$.

The second term of the right side of Eqs. (9.1) and (9.2) are the same. They correspond to the solid phase, that is to say the initial one.

The two Eqs. (9.1) and (9.2) are transcendental. The solution of each one allows obtaining the exact solution of each problem. The solution for each equation is unique and it implies the uniqueness of the similarity solution. In consequence, the Stefan Problem admits only one solution.

The temperature distribution is given by the following equations: (Alexiades and Solomon, 1993)
For the constant temperature boundary condition

$$u^L(x,t) = u_L - u_L \cdot \left(\frac{erf\left(\frac{x}{2\sqrt{\alpha_L t}}\right)}{erf(\lambda)} \right) \tag{10.1. a}$$

$$u^S(x,t) = u_S - u_S \cdot \left(\frac{erfc\left(\frac{x}{2\sqrt{\alpha_S t}}\right)}{erfc(\upsilon\lambda)} \right) \tag{10.1. b}$$

For the flux boundary condition proportional to $t^{-½}$

$$u^L(x,t)=\frac{q_0}{k_L}\cdot\sqrt{\pi\cdot\alpha_L}\cdot\left[erf(\lambda)-erf\left(\frac{x}{2\sqrt{\alpha_L t}}\right)\right] \tag{10.2.a}$$

$$u^S(x,t)=u_S-u_S\cdot\left(\frac{erfc\left(\frac{x}{2\sqrt{\alpha_S t}}\right)}{erfc(\upsilon\lambda)}\right) \tag{10.2.b}$$

The Eqs. (10.1.a) and (10.2.a) are equal to the one obtained for the one phase solution (Boucíguez et al, 2006). The Eq. (10.2.b) is equal to (10.1.b), that is to say the solid phase has the same expression for the two boundary conditions.

In all these equations *erf(ξ)* denotes the error function and *erfc(ξ)* denotes the complementary error function,

$$f(\xi)=erf(\xi)=\frac{2}{\sqrt{\pi}}\int_0^\xi \exp(-\theta^2)\cdot d\theta \tag{11.1}$$

$$erfc(\xi)=1-erf(\xi)=\frac{2}{\sqrt{\pi}}\int_x^\xi \exp(-\theta^2)\cdot d\theta \tag{11.2}$$

The basic properties of these functions are:

$$f(0)=0; \qquad f(\infty)=1$$

$$\frac{df}{d\xi}=f'(\xi)=\frac{2}{\sqrt{\pi}}\exp(-\xi^2)>0 \tag{12}$$

$$\frac{d^2f}{d\xi^2}=f''(\xi)=-2\cdot\xi\cdot f'(\xi)=-\frac{4\xi}{\sqrt{\pi}}\exp(-\xi^2)$$

$$erfc(0)=1; \qquad erfc(\infty)=0$$

RELATIONSHIP BETWEEN THE TWO BOUNDARY CONDITIONS.

It is useful to name the parameter λ in Eqs. (9) and (10), as λ_1 for the temperature condition and λ_2, for the flux condition; so the Eqs. (9) and (10) are written as:

For the temperature boundary condition:

$$\lambda_1\cdot\sqrt{\pi}=\frac{Ste^L}{\exp(\lambda_1^2)\cdot erf(\lambda_1)}-\frac{Ste^S}{\upsilon\exp(\upsilon^2\lambda_1^2)\cdot erf(\upsilon\lambda_1)} \tag{9.1}$$

For the flux boundary condition

$$\lambda_2\cdot\sqrt{\pi}=\frac{q_0}{\rho\cdot L}\sqrt{\frac{\pi}{\alpha_L}}\frac{1}{\exp(\lambda_2^2)}-\frac{Ste^S}{\upsilon\exp(\upsilon^2\lambda_2^2)\cdot erf(\upsilon\lambda_2)} \tag{9.2}$$

as a consequence, Eqs. (10.1) and (10.2) become:

For the temperature boundary condition

$$u^L(x,t)=u_L-u_L\cdot\left(\frac{erf\left(\frac{x}{2\sqrt{\alpha_L t}}\right)}{erf(\lambda_1)}\right) \tag{10.1.a}$$

$$u^S(x,t)=u_S-u_S\cdot\left(\frac{erfc\left(\frac{x}{2\sqrt{\alpha_S t}}\right)}{erfc(\upsilon\lambda_1)}\right) \tag{10.1.b}$$

For the flux boundary condition

$$u^L(x,t)=\frac{q_0}{k_L}\cdot\sqrt{\pi\cdot\alpha_L}\cdot\left[erf(\lambda_2)-erf\left(\frac{x}{2\sqrt{\alpha_L t}}\right)\right] \tag{10.2.a}$$

$$u^S(x,t)=u_S-u_S\cdot\left(\frac{erfc\left(\frac{x}{2\sqrt{\alpha_S t}}\right)}{erfc(\upsilon\lambda_2)}\right) \tag{10.2.b}$$

The equivalence between the two cases will be proved. Evaluating $u^L(0,t)$ from Eq. (10.2.a), results:

$$u^L(0,t)=\frac{q_0}{k_L}\sqrt{\pi\cdot\alpha_L}\cdot erf(\lambda_2) \tag{13}$$

Then $u^L(0,t)$ is constant when the flux is proportional at $t^{-1/2}$, calling u_L to this constant:

$$u_L=\frac{q_0}{k_L}\cdot\sqrt{\pi\alpha_L}\cdot erf(\lambda_2) \tag{14}$$

Taking the derivative of the Eq. (10.1.a) with respect to *x,* and in virtue of the properties of error function, it yields

$$u_x^L(x,t)=-u_L\cdot\frac{1}{\sqrt{\pi\alpha_L}}\cdot\frac{1}{\sqrt{t}}\cdot\exp\left(\frac{-x^2}{4\alpha_L t}\right)\cdot\frac{1}{erf(\lambda_1)} \tag{15}$$

The heat flux at the boundary is $q(t)=-k_L\, u_x(0,t)$, hence evaluating the Eq. (15) at x=0, the heat flux results:

$$q(t)=-k_L u_x(0,t)=k_L u_L \frac{1}{\sqrt{\pi\alpha_L}}\frac{1}{erf(\lambda_1)}\frac{1}{\sqrt{t}}=\frac{cte}{\sqrt{t}} \tag{16}$$

calling q_0 to this constant, results:

$$q_0=\frac{k_L\cdot u_L}{\sqrt{\pi\alpha_L}}\cdot\frac{1}{erf(\lambda_1)} \tag{17}$$

and as a consequence:

$$u_L=\frac{q_0}{k_L}\cdot\sqrt{\pi\alpha_L}\cdot erf(\lambda_1) \tag{18}$$

replacing this value of u_L at Eq. (10.1.a) results:

$$u^L(x,t)=\frac{q_0}{k_L}\sqrt{\pi\alpha_L}\left[erf(\lambda_1)-erf\left(\frac{x}{2\sqrt{\alpha_L t}}\right)\right] \tag{19}$$

This last expression is similar to (10.2.a).

The comparison between Eqs. (14) and (18), shows that, on one hand, they are formally the same and they are equal if and only if $\lambda_1=\lambda_2=\lambda$.

On the other hand, replacing q_0 from Eq. (17) in Eq. (10.2.a) it is obtained:

$$u_x^L(x,t)=-u_L\cdot\frac{1}{erf(\lambda_1)}\cdot\left[erf(\lambda_2)-erf\left(\frac{x}{2\sqrt{\alpha_L t}}\right)\right] \tag{20}$$

This two Eqs. (17) and (10.2.a) are equal if and only if $\lambda_1=\lambda_2=\lambda$.

Thus, it is shown that the two boundary conditions are equivalent.

NUMERICAL EVALUATION

A simple numerical evaluation of the situations discussed in the previous section is presented here, for different values q_0. The results are presented in Table 1; the second column is the value of λ corresponding to Eq. (9.2), that is to say λ_2. The third column is the temperature obtained using this value and Eq. (10.2a). The fourth column is the value of λ corresponding to Eq. (9.1), that is to say λ_1, evaluated with the value of $u(0,t)$ given in the third column.

Table 1. Values of λ_1, λ_2, and boundary temperatures for different values of q_0.

q_0	λ_2	$u(0,t)$	λ_1
100	0,00078215	0,0000981	7,8215 10^{-4}
500	0,0039107	0,002152	3,9107 10^{-3}
1000	0,0078210	0,009809	7,821 10^{-3}
2000	0,0156391	0,039228	1,5639 10^{-2}
3000	0,0234515	0,088228	2,3452 10^{-2}
4000	0,0312554	0,156761	3,1255 10^{-2}
5000	0,0390478	0.244761	3,9048 10^{-2}

In the same way, Table 2 shows, the numerical results for different temperature boundary conditions and the initial one. The first column is the temperature boundary condition, the second the initial one, the third is the λ value obtained of Eq. (9.1), that is to say λ_1. The fourth column is the obtained value of q_0 (W $s^{1/2}/m^2$) using this λ value. The fifth column is the λ value evaluated using the obtained value of q_0 and the initial temperature using Eq. (9.2), that is to say λ_2.

Table 2. Values of λ_1, λ_2, q_0 for different values of temperatures boundary and initial conditions.

u_L	u_0	λ_1	q_0	λ_2
20	–2	0.32659	1.2626 10^4	0.32659
20	–5	0.35419	1.1713 10^4	0.35419
25	–2	0.35788	1.4502 10^4	0.35788
25	–5	0.38259	1.3646 10^4	0.38259
30	–2	0.38589	1.6248 10^4	0.38589
30	–5	0.40834	1.5442 10^4	0.40834

All these calculations where made using the Mathematical Support of the Scientific Work Place, so the precision is ensured. The results showed in both tables allow to say that the two values λ_1 and λ_2, obtained by different ways are equal, as it was proved in the last section.

DISCUSSION AND CONCLUSION

The analysis of Figures 2 to 5, for a particular material (organic wax), shows that the heat flux $q_0/t^{1/2}$ produces a constant temperature in the fixed face. The obtained results are summarized in Table 3. The first column is the value of q_0, the second one is the obtained result for $u(0,t)$ from Eq. (10.2.a), using this q_0, and the corresponding value of λ_2. The third column is the relationship between that q_0 and the minor value (q_0=100), that is to say, it was obtained taking 100, as the first value. Thus, for q_0=500, this factor is 500/100=5, for q_0=1000, is 1000/100=10 and so on. The fourth column is the ratio between $u(0,t)$ and the value of $u(0,t)$ corresponding to q_0=100. Finally the fifth column is the ratio between the fourth and second columns.

Table 3. q_0 values and the corresponding boundary temperature.

$q0$	$u(0,t)$ C			
100	0.00010	1	1	1
500	0.0025	5	25	5
1000	0.010	10	100	10
2000	0.04	20	400	20

These results together with those show in Tab. 1 and 2, validate the equivalence between the two boundary conditions.

Table 3 also shows that when q_0 increases, the temperature in the fixed face is multiplied by the squared ratio between the new and the old q_0 (third column)

This ratio only depends on q_0, given that it is the only factor that changes in Eq. (10.2) for different materials. Thus, this relationship is also valid for all substances.

Thus, it is proved that the two well known analytical solutions for the two - phases Stefan Problem, one for temperature boundary condition, the other for heat flux condition are not independent: one determines the other.

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