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Article

OUTLIER DETECTION IN PARTIAL ERRORS-IN-VARIABLES MODEL

Detecção de outliers usando um modelo de observações de erro

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Abstract:
The weighed total least square (WTLS) estimate is very sensitive to the outliers in the partial EIV model. A new procedure for detecting outliers based on the data-snooping is presented in this paper. Firstly, a two-step iterated method of computing the WTLS estimates for the partial EIV model based on the standard LS theory is proposed. Secondly, the corresponding *w*-test statistics are constructed to detect outliers while the observations and coefficient matrix are contaminated with outliers, and a specific algorithm for detecting outliers is suggested. When the variance factor is unknown, it may be estimated by the least median squares (LMS) method. At last, the simulated data and real data about two-dimensional affine transformation are analyzed. The numerical results show that the new test procedure is able to judge that the outliers locate in x component, y component or both components in coordinates while the observations and coefficient matrix are contaminated with outliers.
Keywords: Partial EIV model; Two-step iterated method; Weighted total least-squares; Outlier detection; Data-snooping; Two-dimensional affine transformation

Resumo:

O estimador dos Mínimos Quadrados Total é muito sensível à presença de outliers no modelo de observações de erro. Neste trabalho apresenta-se um novo modelo para detecção de outliers baseado na técnica *data-snooping*. Primeiro, é proposto um método iterativo para determinar o estimador dos Mínimos Quadrados Total na teoria dos Mínimos Quadrados. Em seguida, o teste estatatístico w é construído para detectar outliers enquanto as observações e a matriz de coeficientes são contaminadas com a presença de outliers, sendo sugerido um algoritmo específico para detecção de outliers. Quando o fator de variância é desconhecido, ele deve ser estimado pelo método dos Mínimos Quadrados Medianos. Foram analisados dados simulados e reais. Os resultados numéricos mostraram que o método proposto é capaz de identificar se os

outliers se encontram nas componentes em x ou em y, enquanto as observações e a matriz de coeficientes são contaminados com outliers.

Palavras-chave: Modelo EIV Parcial; Método Iterativo Two-step; Estimador dos Mínimos quadrados Total; Detecção de Outlier; Data-snooping; Transformação Afim bidimensional.

1. INTRODUCTION

Gauss-Markov (G-M) model and least-squares (LS) method are widely used in geodetic science. Most of time, the elements of the coefficient matrix may be consisting of the observations possessing the statistical properties in many applications such as the coordinate transformation (Akyilmaz, 2007; Li et al., 2012; Li et al., 2013; Fang, 2014), and the estimates of the unknown parameters derived by the LS method would not be optimal because the statistical properties of the elements in the coefficient matrix are ignored. The errors-in-variables (EIV) model and so called total least-squares (TLS) method named by Gloub et al. (1980) are more rigorous than the LS method. There are many algorithms to compute the TLS estimate (Gloub et al.,1980; Schaffrin, 2006) or weighted TLS (WTLS) estimate (Schaffrin and Wieser, 2008; Shen et al., 2011; Xu et al., 2012; Amiri-Simkooei and Jazaeri, 2012; Mahboub, 2012; Fang, 2013; Jazaeri et al., 2014).

Unfortunately, like the LS estimate, the WTLS estimate is also extremely vulnerable to the outliers in the EIV model. Although many methods for detecting the outliesr in the G-M model are investigated extensively (Baarda, 1968; Pope, 1976; Kok, 1984; Huber 1981; Hekimoglu, 2005; Gui et al. 1999, 2005a, 2005b, 2007, 2011; Guo et al., 2007; Hekimoglu and Erenoglu, 2009; Lehmann, 2013; Hekimoglu et al., 2014), they cannot be directly employed to deal with the outliers in the EIV model. Schaffrin and Uzun (2011) have generalized the mean-shift method to detect a single outlier located either in the observations or in the coefficient matrix in the EIV model. The reliability was also analyzed (Schaffrin and Uzun, 2012). Amiri-Simkooei and Jazaeri (2013) applied the data-snooping procedure to identify the outliers based on the WTLS method formulated with the standard LS theory (Amiri-Simkooei and Jazaeri, 2012). However, the test procedure is required to be implemented more than once while there are some repeated random elements in the different locations of the coefficient matrix like the two-dimensional affine transformation.

The partial EIV model is a generalized EIV model and can avoid considering the correlations between the repeated random elements in the coefficient matrix (Xu et al., 2012). Therefore, it is a more proper model to be used to deal with the case where the coefficient matrix follows a structured characteristic. Unfortunately, the test statistics for detecting the outliers cannot be clearly derived through the existing WTLS method. For this reason, a new two-step iterated approach of computing the WTLS estimates under the framework of LS theory is developed in this paper so that some test statistics of identifying the outliers for the partial EIV model can be constructed.

The remaining of the paper is organized as follows. In Section 2, a two-step iterated method for the partial EIV model taking advantage of LS theory is proposed. In Section 3, the corresponding w-test statistics are constructed to detect the outliers while the observations, coefficient matrix or both are contaminated with the outliers and an algorithm for detecting outliers in the partial EIV model is designed. If the variance factor is not known, we will employ the least median squares (LMS) method to estimate it. In a latter section, a simulated data and a real data about two-dimensional affine transformation are used to verify the validity of the proposed method. In the end, some concluding remarks are presented.

2. PARTIAL EIV MODEL AND WTLS ESTIMATE

As a matter of fact, not all elements of the coefficient matrix are random and there are some repeated random elements in the different locations of the coefficient matrix such as the coordinate transformation. As a result, their correlations between the repeated random elements must be taken into account. The five rules (Mahboub, 2012) can be used to determine the variance-covariance matrix of the coefficient matrix. However, if the partial EIV model proposed by Xu et al. (2012) is considered, the correlations can be avoided so that the additional burden is reduced. Therefore, the partial EIV model is more superior to be adopted. The function model is shown as following:

$$\begin{cases} \boldsymbol{L} = \left(\boldsymbol{X}^T \otimes \boldsymbol{I}_n\right)\left(\boldsymbol{h} + \boldsymbol{B}\bar{\boldsymbol{a}}\right) + \boldsymbol{\Delta} \\ \boldsymbol{a} = \bar{\boldsymbol{a}} + \boldsymbol{e} \end{cases} \tag{1}$$

Where $\boldsymbol{X}$= t×1 vector of unknown parameters; $\boldsymbol{L}$= n×1 vector of observations; $\boldsymbol{I}_n$= n×n identity matrix; $\boldsymbol{h}$= nt×1 vector that is consisting of zero and fixed elements of the coefficient matrix $\boldsymbol{A}$;$\boldsymbol{B}$ =nt×s known structured matrix; s=the number of different random elements of $\boldsymbol{A}$ = $invec(\boldsymbol{h}+\boldsymbol{Ba})$; $\bar{\boldsymbol{a}}$ = s ×1 true values vector of $\boldsymbol{a}$; $\boldsymbol{e}$ = s ×1 random errors vector of $\boldsymbol{a}$; $\boldsymbol{\Delta}$= n ×1 vector of random errors of observations; *invec* is a mathematic function for transforming an nt×1 vector to an n × t matrix; $\otimes$ =Kronecker product operator. The stochastic model is expressed as follows:

$$\begin{bmatrix} \boldsymbol{\Delta} \\ \boldsymbol{e} \end{bmatrix} \sim N_{n+s}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma^2 \begin{bmatrix} \boldsymbol{Q}_L & 0 \\ 0 & \boldsymbol{Q}_a \end{bmatrix}\right) \tag{2}$$

Where $\boldsymbol{Q}_L$= n×n cofactor matrix of $\boldsymbol{L}$; $\boldsymbol{Q}_a$= s×s cofactor matrix of $\boldsymbol{a}$; σ^2=unknown variance factor. A two-step iterated method of computing the WTLS estimate for the partial EIV model is proposed in order to develop an outlier detection method suitable for the partial EIV model. For any given $\boldsymbol{X}^{(0)}$, the model (1) can be transformed as follows:

$$\begin{cases} \boldsymbol{L} - \left[\left(\boldsymbol{X}^{(0)}\right)^T \otimes \boldsymbol{I}_n\right]\boldsymbol{h} = \left[\left(\boldsymbol{X}^{(0)}\right)^T \otimes \boldsymbol{I}_n\right]\boldsymbol{B}\bar{\boldsymbol{a}} + \boldsymbol{\Delta} \\ \boldsymbol{a} = \bar{\boldsymbol{a}} + \boldsymbol{e} \end{cases} \tag{3}$$

Furthermore, the model (3) can be rewritten as

$$\begin{bmatrix} \bar{\boldsymbol{L}} \\ \boldsymbol{a} \end{bmatrix} = \begin{bmatrix} \left(\left(\boldsymbol{X}^{(0)}\right)^{T} \otimes \boldsymbol{I}_n\right)\boldsymbol{B} \\ \boldsymbol{I}_s \end{bmatrix} \bar{\boldsymbol{a}} + \begin{bmatrix} \boldsymbol{\Delta} \\ \boldsymbol{e} \end{bmatrix} \tag{4}$$

where $\bar{\boldsymbol{L}} = \boldsymbol{L} - \left(\left(\boldsymbol{X}^{(0)}\right)^{T} \otimes \boldsymbol{I}_n\right)\boldsymbol{h}$. If we denote

$$\widehat{\boldsymbol{L}} = \begin{bmatrix} \bar{\boldsymbol{L}} \\ \boldsymbol{a} \end{bmatrix}, \widehat{\boldsymbol{A}} = \begin{bmatrix} \left(\left(\boldsymbol{X}^{(0)}\right)^{T} \otimes \boldsymbol{I}_n\right)\boldsymbol{B} \\ \boldsymbol{I}_s \end{bmatrix}, cov\begin{bmatrix} \boldsymbol{\Delta} \\ \boldsymbol{e} \end{bmatrix} = \sigma^2 \boldsymbol{Q}_{\widehat{L}} = \sigma^2 \begin{bmatrix} \boldsymbol{Q}_L & 0 \\ 0 & \boldsymbol{Q}_a \end{bmatrix} \tag{5}$$

the estimate of $\bar{\boldsymbol{a}}$ can be derived by the LS principle (Koch, 1999). As a result, we have

$$\hat{\bar{\boldsymbol{a}}} = \left(\widehat{\boldsymbol{A}}^T \boldsymbol{Q}_{\widehat{L}}^{-1} \widehat{\boldsymbol{A}}\right)^{-1} \widehat{\boldsymbol{A}}^T \boldsymbol{Q}_{\widehat{L}}^{-1} \widehat{\boldsymbol{L}} \tag{6}$$

The residual vector of $\boldsymbol{a}$ is

$$\boldsymbol{V}_a = \boldsymbol{a} - \hat{\bar{\boldsymbol{a}}} \tag{7}$$

Inserting $\hat{\bar{\boldsymbol{a}}}$ into the first equation of the model (1) yields

$$\boldsymbol{L} = \left(\boldsymbol{X}^{\boldsymbol{T}} \otimes \boldsymbol{I}_n\right)\left(\boldsymbol{h} + \boldsymbol{B}\hat{\bar{\boldsymbol{a}}}\right) + \boldsymbol{\Delta} \tag{8}$$

If the inverse transformation of the mathematic operator *vec* (*invec*) is used, we can obtain

$$\bar{\boldsymbol{A}} = invvec\left(\boldsymbol{h} + \boldsymbol{B}\hat{\bar{\boldsymbol{a}}}\right) \tag{9}$$

Then the model (8) is easily rewritten as follows:

$$\boldsymbol{L} = \bar{\boldsymbol{A}}\boldsymbol{X} + \boldsymbol{\Delta} \tag{10}$$

Similarly, based on the LS principle (Koch, 1999), the estimate of $\boldsymbol{X}$ is

$$\hat{\boldsymbol{X}} = \left(\bar{\boldsymbol{A}}^T \boldsymbol{Q}_L^{-1} \bar{\boldsymbol{A}}\right)^{-1} \bar{\boldsymbol{A}}^T \boldsymbol{Q}_L^{-1} \boldsymbol{L} \tag{11}$$

and the residual vector of $\boldsymbol{L}$ is

$$\boldsymbol{V}_L = \boldsymbol{L} - \bar{\boldsymbol{A}}\hat{\boldsymbol{X}} \tag{12}$$

The posterior estimate of the variance factor, which can be obtained from Equation 7 and Equation 12, is

$$\hat{\sigma}^2 = \frac{\boldsymbol{V}_L{}^T \boldsymbol{Q}_L^{-1} \boldsymbol{V}_L + \boldsymbol{V}_a{}^T \boldsymbol{Q}_a^{-1} \boldsymbol{V}_a}{n-t} \tag{13}$$

3. OUTLIER DETECTION PROCEDURE IN PARTIAL EIV

MODEL

The data-snooping method suggested by Baarda (1968) is employed extensively in geodetic data processing for detecting the outliers (Kok, 1984; Koch, 1999). If the observations or coefficient matrix in the partial EIV model are contaminated with the outliers, the following *w*-test statistics can be constructed based on Equation 6 or Equation 11 to detect the outliers:

$$w_{ai} = \frac{\boldsymbol{g}_i^T \boldsymbol{Q}_{\hat{L}}^{-1} \boldsymbol{V}_{\hat{L}}}{\sigma\sqrt{\boldsymbol{g}_i^T \boldsymbol{Q}_{\hat{L}}^{-1} \boldsymbol{R}_{\hat{L}} \boldsymbol{g}_i}} \sim N(0,1) \tag{14}$$

$$w_{Lj} = \frac{\boldsymbol{f}_j^T \boldsymbol{Q}_L^{-1} \boldsymbol{V}_L}{\sigma\sqrt{\boldsymbol{f}_j^T \boldsymbol{Q}_L^{-1} \boldsymbol{R}_L \boldsymbol{f}_j}} \sim N(0,1) \tag{15}$$

where $\boldsymbol{V}_{\hat{L}} = \hat{\boldsymbol{L}} - \hat{\boldsymbol{A}}\hat{\hat{\boldsymbol{a}}}$, $\boldsymbol{R}_{\hat{L}} = \boldsymbol{I}_{n+s} - \hat{\boldsymbol{A}}\left(\hat{\boldsymbol{A}}^T \boldsymbol{Q}_{\hat{L}}^{-1}\hat{\boldsymbol{A}}\right)^{-1} \hat{\boldsymbol{A}}^T \boldsymbol{Q}_{\hat{L}}^{-1}$, $\boldsymbol{R}_L = \boldsymbol{I}_n - \bar{\boldsymbol{A}}\left(\bar{\boldsymbol{A}}^T \boldsymbol{Q}_L^{-1}\bar{\boldsymbol{A}}\right)^{-1} \bar{\boldsymbol{A}}^T \boldsymbol{Q}_L^{-1}$; $\boldsymbol{g}_i^T = [0,\cdots,1,\cdots 0]_{(n+s)\times 1}$ and $\boldsymbol{f}_j^T = [0,\cdots,1,\cdots 0]_{n\times 1}$ are an unit vector with the *i*th and *j*th element equal to 1, respectively; *N*(0,1) represents the standard normal distribution.

In general, when the variance factor is unknown, its posterior estimate $\hat{\sigma}^2$ can be adopted (Pope, 1976). Then we have

$$\tilde{w}_{ai} = \frac{\boldsymbol{g}_i^T \boldsymbol{Q}_{\hat{L}}^{-1} \boldsymbol{V}}{\hat{\sigma}\sqrt{\boldsymbol{g}_i^T \boldsymbol{Q}_{\hat{L}}^{-1} \boldsymbol{R}\boldsymbol{g}_i}} \sim \tau_n \tag{16}$$

and

$$\tilde{w}_{Lj} = \frac{\boldsymbol{f}_j^T \boldsymbol{Q}_L^{-1} \boldsymbol{V}_L}{\hat{\sigma}\sqrt{\boldsymbol{f}_j^T \boldsymbol{Q}_L^{-1} \boldsymbol{R}_L \boldsymbol{f}_j}} \sim \tau_{n-t} \tag{17}$$

Where τn = τ distribution with n degree of freedom. The computation about τ distribution can be found in Baselga (2007) and Guo and Zhao (2012).

The robust method is an efficient one to estimate the variance factor. By employing the least median squares (LMS) method (Rousseeuw and Leroy, 1987), the variance factor may be estimated by

$$\hat{\sigma}_a = 1.4826\sqrt{\text{median}\left(\sigma^2 w_{ai}^2\right)} \tag{18}$$

or

$$\hat{\sigma}_L = 1.4826\sqrt{\text{median}\left(\sigma^2 w_{Lj}^2\right)} \tag{19}$$

So the test statistics (14) and (15) with (18) and (19) become

$$\bar{w}_{ai} = \frac{\boldsymbol{g}_i^T \boldsymbol{Q}_{\hat{L}}^{-1} \boldsymbol{V}}{\hat{\sigma}_a \sqrt{\boldsymbol{g}_i^T \boldsymbol{Q}_{\hat{L}}^{-1} \boldsymbol{R} \boldsymbol{g}_i}} \tag{20}$$

and

$$\bar{w}_{Lj} = \frac{\boldsymbol{f}_j^T \boldsymbol{Q}_L^{-1} \boldsymbol{V}_L}{\hat{\sigma}_L \sqrt{\boldsymbol{f}_j^T \boldsymbol{Q}_L^{-1} \boldsymbol{R}_L \boldsymbol{f}_j}} \tag{21}$$

The superiority of the above two test statistics is that they are very robust to the outliers so that it is more reliable for them to be used for detecting the outliers. It is to be noted here that they do not strictly follow a normal distribution. Therefore, it is very hard to give the exact probability distributions of them. In order to simplify the computation of the threshold value which is used to identify the outliers, the upper percentage point of the standard normal distribution is still used when the principle of identifying the outliers is established.

The implemented procedure for detecting the outliers in the partial EIV model is summarized as follows:

Step1. Give $\boldsymbol{a}, \boldsymbol{L}, \boldsymbol{h}, \boldsymbol{B}, \boldsymbol{Q}_L, \boldsymbol{Q}e$ and define $\boldsymbol{Q}_{\hat{L}} = \begin{bmatrix} \boldsymbol{Q}_L & 0 \\ 0 & \boldsymbol{Q}_e \end{bmatrix}$.

Step2. Set the initial value $\hat{\boldsymbol{X}}^{(0)} = \left(\boldsymbol{A}^T \boldsymbol{Q}_L^{-1} \boldsymbol{A}\right)^{-1} \boldsymbol{A}^T \boldsymbol{Q}_L^{-1} \boldsymbol{L}$.

Step3. For any k, compute

$$\bar{\boldsymbol{L}}^{(k)} = \boldsymbol{L} - \left(\left(\boldsymbol{X}^{(k)}\right)^T \otimes \boldsymbol{I}_n\right)\boldsymbol{h}, \ \hat{\boldsymbol{L}}^{(k)} = \begin{bmatrix} \bar{\boldsymbol{L}}^{(k)} \\ \boldsymbol{a} \end{bmatrix}, \ \hat{\boldsymbol{A}}^{(k)} = \begin{bmatrix} \left(\left(\boldsymbol{X}^{(k)}\right)^T \otimes \boldsymbol{I}_n\right)\boldsymbol{B} \\ \boldsymbol{I}_t \end{bmatrix}.$$

Step4. Compute $\hat{\bar{\boldsymbol{a}}}^{(k)} = \left(\left(\hat{\boldsymbol{A}}^{(k)}\right)^T \boldsymbol{Q}_{\hat{L}}^{-1} \hat{\boldsymbol{A}}^{(k)}\right)^{-1} \left(\hat{\boldsymbol{A}}^{(k)}\right)^T \boldsymbol{Q}_{\hat{L}}^{-1} \hat{\boldsymbol{L}}^{(k)}$.

Step5. Compute $\bar{\boldsymbol{A}}^{(k)} = invvec\left(\boldsymbol{h} + \boldsymbol{B}\hat{\bar{\boldsymbol{a}}}^{(k)}\right)$ and $\hat{\boldsymbol{X}}^{(k+1)} = \left(\left(\bar{\boldsymbol{A}}^{(k)}\right)^T \boldsymbol{Q}_L^{-1} \bar{\boldsymbol{A}}^{(k)}\right)^{-1} \left(\bar{\boldsymbol{A}}^{(k)}\right)^T \boldsymbol{Q}_L^{-1} \boldsymbol{L}$.

Step6. If $\left\|\hat{\boldsymbol{X}}^{(k+1)} - \hat{\boldsymbol{X}}^{(k)}\right\| < \varepsilon$, the iteration will be stopped, where ε is a given value. Otherwise, return to Step 3.

Step7. Compute $\boldsymbol{V}_a^{(k)} = \boldsymbol{a} - \hat{\bar{\boldsymbol{a}}}^{(k)}, \boldsymbol{V}_L^{(k)} = \boldsymbol{L} - \boldsymbol{A}\bar{\boldsymbol{X}}^{(k+1)}$ and

$$\hat{\sigma}^2 = \frac{\left(\boldsymbol{V}_L^{(k)}\right)^T \boldsymbol{Q}_L^{-1} \boldsymbol{V}_L^{(k)} + \left(\boldsymbol{V}_a^{(k)}\right)^T \boldsymbol{Q}_a^{-1} \boldsymbol{V}_a^{(k)}}{n-t}.$$

Step8. According to the data-snooping procedure, for single outlier, if

$$\left|\bar{w}_{ai}^{(k)}\right| > u_{1-\alpha/2} \quad \left(i \in \left(n+1,\cdots,n+s\right)\right)$$

and

$$\left|\bar{w}_{Lj}^{(k)}\right| > u_{1-\alpha/2} \quad \left(j \in \left(1,\cdots,n\right)\right)$$

are satisfied simultaneously, one can judge that the outlier locates in the observation equation containing the observation L_j and coefficient matrix element $\boldsymbol{a}_i$. For multiple outliers, if

$$\max_{i\in\{n+1,\cdots,n+s\}} \left|\bar{w}_{ai}^{(k)}\right| > u_{1-\alpha/2}$$

and

$$\max_{j\in\{1,\cdots,n\}} \left|\bar{w}_{Lj}^{(k)}\right| > u_{1-\alpha/2}$$

we will deem that the corresponding observation equation containing the observations Lj and coefficient matrix elements ai is contaminated with outlier. But one still can't confirm that the outliers locate in the observations or coefficient matrix, or both. Here uα is the upper α-percentage point of the standard normal distribution.

Step 9. If multiple outliers exist in the observations or coefficient matrix, the above procedure of Step 1 to Step 8 should be repeated until all the w-test statistics are smaller than the threshold value.

4. NUMERICAL RESULTS

4.1. Simulated two-dimensional affine transformation

The mathematic model for the two-dimensional affine transformation is expressed as follows:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} x_s & y_s & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_s & y_s & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix} \tag{22}$$

Table 1: Coordinates of points with random errors in start system and target system (unit: m)

coordinate	1	2	3	4	5	6	7	8	9	10
x_s	70.00	66.16	56.17	43.83	33.82	30.00	33.80	43.83	56.17	66.19
y_s	49.98	61.74	69.02	69.01	61.77	50.00	38.25	30.97	30.98	38.24
x_t	180.00	141.21	86.70	37.26	11.77	19.99	58.77	113.31	162.77	188.24
y_t	59.98	114.67	163.71	188.45	179.38	140.00	85.35	36.28	11.56	20.61

The data are displayed in Table 1, which is taken from Amiri-Simkooei and Jazaeri (2013). In this example, there are ten points in total. So the partial EIV model is

$$\begin{cases} \boldsymbol{L} = \left(\boldsymbol{X}^T \otimes \boldsymbol{I}_{20}\right)\left(\boldsymbol{h} + \boldsymbol{B}\bar{\boldsymbol{a}}\right) + \Delta \\ \boldsymbol{a} = \bar{\boldsymbol{a}} + \boldsymbol{e} \end{cases} \tag{23}$$

where $\boldsymbol{a} = \left[x_s^1, y_s^1, x_s^2, y_s^2 \ \cdots \ x_s^{10}, y_s^{10}\right]^T$, $\boldsymbol{h}^T = \left[\boldsymbol{h}_1^T \ \boldsymbol{h}_2^T \ \cdots \ \boldsymbol{h}_6^T\right]$, $\boldsymbol{h}_1^T = \boldsymbol{h}_2^T = \boldsymbol{h}_4^T = \boldsymbol{h}_5^T = \left[0, 0, \cdots, 0\right]_{1\times 20}$,

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \\ \vdots \\ \boldsymbol{B}_6 \end{bmatrix}, \boldsymbol{B}_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{20\times 20}, \boldsymbol{B}_2 = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{20\times 20}, \boldsymbol{B}_3 = \boldsymbol{B}_6 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{20\times 20}$$

$$\boldsymbol{B}_4 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{20\times 20}, \boldsymbol{B}_5 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{20\times 20}.$$

In order to give the reliable evaluations for the proposed outlier detection method, the following five schemes for adding outliers are discussed. The significant level for determining critical value is set as 0.05, which is very frequently used (Gao et al. 1992).

Scheme 1: According to Amiri-Simkooei and Jazaeri (2013), the outlier of magnitude 0.1 m which is 10 times of the priori standard deviation, is added into the xs component of point 4 in the start system.

The residuals of the observations and random vector a and the corresponding w-test statistics are displayed in Table 2. Obviously, the absolute values of residuals of the x components of point 4 in the start system and target system are greater than others. Meanwhile, both $\left|\bar{w}_{a27}\right| = 4.6774$ and $\left|\bar{w}_{L7}\right| = 3.7011$ surpass the threshold value u0.975 = 1.96. So we deem that there is an outlier in the x component of the start system, target system or both, which is kept the same with the set simulated case. However, we can't determinate the special position of the outlier.Scheme 2: The outlier of magnitude 0.1 m is added into both components of point 4 in the start system.The residuals and w-test statistics are shown in Table 3. As we know, the absolute values of residuals of the x components of point 4 in both coordinate systems are greater than others.Particularly, both $\left|\bar{w}_{a27}\right| = 2.7943$ and $\left|\bar{w}_{L7}\right| = 3.2089$ for the x component of point 4 are beyond the threshold value 1.96, and $\left|\bar{w}_{L8}\right| = 2.8142$ for the yt component of point 4 in the target system exceeds 1.96 too. Although $\left|\bar{w}_{a28}\right| = 1.8708$ for the ys component of point 4 in the start system is smaller than the threshold value 1.96, the absolute values of w-test statistics and their corresponding absolute values of residuals are very tremendous. Thus, both components of point 4 are considered to be contaminated with outliers. Unfortunately, we can't discriminate the specific positions of these outliers.

Table 2: Residuals of observations and random vector ***a*** and corresponding *w*-test statistics (Scheme 1) (unit: m)

Point No.	Target system			Start system		
	Coordinate	$\boldsymbol{V}_{Lj}$	$\overline{w}_{Lj}$	Coordinate	$\boldsymbol{V}_{ai}$	$\overline{w}_{ai}$
1	x_t	-0.0013973	-0.25266	x_s	0.0048841	0.32287
	y_t	-0.00035329	-0.063881	y_s	-0.0013852	-0.091561
2	x_t	0.0091772	1.6594	x_s	-0.025934	-1.7144
	y_t	0.0053935	0.97525	y_s	-0.025934	-0.21159
3	x_t	0.008246	1.491	x_s	-0.029849	-1.9732
	y_t	0.0015716	0.28417	y_s	0.010229	0.67613
4	x_t	**-0.020466**	**-3.7011**	x_s	**0.070755**	**4.6774**
	y_t	-0.0055651	-1.0064	y_s	-0.018725	-1.2377
5	x_t	0.0046209	0.83561	x_s	-0.020742	-1.3712
	y_t	-0.0011276	-0.20392	y_s	0.013769	0.91015
6	x_t	0.0036126	0.65332	x_s	-0.010178	-0.67285
	y_t	0.0021386	0.38674	y_s	-0.0013217	-0.087364
7	x_t	0.0054174	0.97969	x_s	-0.017635	-1.1658
	y_t	0.0020203	0.36536	y_s	0.0027668	0.18289
8	x_t	-0.0040845	-0.7386	x_s	0.016878	1.1158
	y_t	0.00026835	0.048526	y_s	-0.0092558	-0.61182
9	x_t	0.00051964	0.093964	x_s	-0.0030844	-0.2039
	y_t	-0.00050291	-0.090939	y_s	0.0030535	0.20184
10	x_t	-0.0056462	-1.021	x_s	0.014906	0.98537
	y_t	-0.0038434	-0.69502	y_s	0.0040706	0.26907

Table 3: Residuals of observations and random vector ***a*** and corresponding *w*-test statistics (Scheme 2) (unit: m)

Point No.	Target system			Start system		
	Coordinate	$\boldsymbol{V}_{Lj}$	$\overline{w}_{Lj}$	Coordinate	$\boldsymbol{V}_{ai}$	$\overline{w}_{ai}$
1	x_t	-0.00093979	-0.10477	x_s	0.0052099	0.22502
	y_t	0.00072597	0.080929	y_s	-0.0047832	-0.20659
2	x_t	0.011106	1.238	x_s	-0.024541	-1.06
	y_t	0.0099526	1.1094	y_s	-0.017565	-0.75868
3	x_t	0.011362	1.2665	x_s	-0.024541	-1.1918
	y_t	0.0089374	0.99627	y_s	-0.017565	-0.56127
4	x_t	**-0.028779**	**-3.2089**	x_s	**0.064696**	**2.7943**
	y_t	**-0.025239**	**-2.8142**	y_s	**0.043314**	**1.8708**
5	x_t	0.0077303	0.86182	x_s	-0.018478	-0.79809
	y_t	0.0062289	0.69444	y_s	-0.0094334	-0.40745
6	x_t	0.0055308	0.61665	x_s	-0.0087742	-0.37897
	y_t	0.006682	0.745	y_s	-0.015647	-0.6758
7	x_t	0.0058653	0.65393	x_s	-0.017298	-0.7471
	y_t	0.0030857	0.34403	y_s	-0.00059882	-0.025864
8	x_t	-0.0048241	-0.5378	x_s	0.016352	0.70624
	y_t	-0.0014746	-0.16439	y_s	-0.0037588	-0.16235
9	x_t	-0.00067131	-0.074838	x_s	-0.0039448	-0.17038
	y_t	-0.0033185	-0.36994	y_s	0.011924	0.51502
10	x_t	-0.0063798	-0.71127	x_s	0.014371	0.62072
	y_t	-0.0055804	-0.62215	y_s	0.0095432	0.41219

Scheme 3: The outlier of magnitude 0.1 m is added into the x_s component of point 4 in the start system and the y_t component of point 4 in target system.

The residuals and the *w*-test statistics are obtained, which is displayed in Table 4. The results from Table 4 show that the test statistics satisfy $|\overline{w}_{a27}| = 4.9415 > 1.96$ and $|\overline{w}_{L7}| = 4.106 > 1.96$, which shows that the *x* component of point 4 is possibly contaminated with an outlier. Although the absolute value of residual for the y_t component of point 4 in the target system is small, $|\overline{w}_{a28}| = 3.0386 > 1.96$ and the absolute value of residual for the y_s component of point 4 in the start system demonstrate that there is an outlier in the *y* component.

Table 4: Residuals of observations and random vector ***a*** and corresponding *w*-test statistics (Scheme3) (unit: m)

Point No.	Target system			Start system		
	Coordinate	$\boldsymbol{V}_{Lj}$	$\overline{w}_{Lj}$	Coordinate	$\boldsymbol{V}_{ai}$	$\overline{w}_{ai}$
1	x_t y_t	-0.0017266 -0.00078794	-0.49177 -0.22442	x_s y_s	0.0053326 -0.00030473	0.42223 -0.024126
2	x_t y_t	0.0077803 0.0035539	2.2158 1.0121	x_s y_s	-0.024022 0.0013597	-1.9021 0.10765
3	x_t y_t	0.0059859 -0.0014011	1.7047 -0.399	x_s y_s	-0.02675 0.017599	-2.1181 1.3933
4	x_t y_t	**-0.014411** **0.0023763**	**-4.106** **0.67706**	x_s y_s	**0.062409** **-0.038379**	**4.9415** **-3.0386**
5	x_t y_t	0.002359 -0.0040971	0.67191 -1.167	x_s y_s	-0.01763 0.021125	-1.396 1.6725
6	x_t y_t	0.0022133 0.00030441	0.63048 0.086714	x_s y_s	-0.0082465 0.0032151	-0.65296 0.25455
7	x_t y_t	0.0050847 0.001589	1.4484 0.45262	x_s y_s	-0.017166 0.0038252	-1.3592 0.30285
8	x_t y_t	-0.0035534 0.00097174	-1.0121 0.27677	x_s y_s	0.01616 -0.011008	1.2795 -0.8715
9	x_t y_t	0.0013812 0.00063268	0.39339 0.1802	x_s y_s	-0.004261 0.00023425	-0.33738 0.018546
10	x_t y_t	-0.0051132 -0.0031418	0.39339 0.1802	x_s y_s	0.014175 0.0023335	1.1224 0.18475

Table 5: Residuals of observations and random vector ***a*** and corresponding *w*-test statistics (Scheme 4) (unit: m)

Point No.	Target system			Start system		
	Coordinate	$\boldsymbol{V}_{Lj}$	$\overline{w}_{Lj}$	Coordinate	$\boldsymbol{V}_{ai}$	$\overline{w}_{ai}$
1	x_t y_t	-0.0023489 -0.00016403	-0.58721 -0.041006	x_s y_s	0.0090658 -0.0040362	0.75905 -0.33795
2	x_t y_t	0.0051497 0.00618	1.2873 1.5449	x_s y_s	-0.0082485 -0.014416	-0.69062 -1.207
3	x_t y_t	0.0017338 0.0028432	0.4334 0.71071	x_s y_s	-0.0012538 -0.0079015	-0.10498 -0.6616
4	x_t y_t	-0.0030607 **-0.008957**	-0.76547 **-2.2401**	x_s y_s	-0.0056531 **0.029688**	-0.47332 **2.4858**
5	x_t y_t	-0.0018874 0.00014316	-0.47184 0.035789	x_s y_s	0.0078336 -0.0043421	0.65589 -0.36357
6	x_t y_t	-0.00040699 0.0029242	-0.10176 0.73114	x_s y_s	0.0074691 -0.012501	0.62537 -1.0467
7	x_t y_t	0.0044704 0.0021985	1.1177 0.54968	x_s y_s	-0.013485 0.00014176	-1.1291 0.01187
8	x_t y_t	-0.0025448 -3.0876e-005	-0.63621 -0.0077191	x_s y_s	0.010115 -0.0049597	0.84692 -0.41528
9	x_t y_t	0.0030058 -0.00099484	0.75143 -0.2487	x_s y_s	-0.014008 0.0099803	-1.1728 0.83567
10	x_t y_t	-0.0041109 -0.0041423	-1.0278 -1.0356	x_s y_s	0.0081649 0.0083459	0.68362 0.69881

Table 6: Residuals of observations and random vector ***a*** and corresponding *w*-test statistics with deleting point 4 in both of start system and target system (unit: m)

Point No.	Target system			Start system		
	Coordinate	$\boldsymbol{V}_{Lj}$	$\bar{w}_{Lj}$	Coordinate	$\boldsymbol{V}_{ai}$	$\bar{w}_{ai}$
1	x_t y_t	-0.0025167 -0.00065539	-0.7071 -0.18414	x_s y_s	0.0087558 -0.0024088	0.89974 -0.24753
2	x_t y_t	0.0044403 0.0041034	1.2804 1.1833	x_s y_s	-0.0095588 -0.0075354	-0.98226 -0.77434
3	x_t y_t	0.0005872 -0.00051201	0.17764 -0.15489	x_s y_s	-0.0033716 0.0032211	-0.34646 0.331
4	x_t y_t	— —	— —	x_s y_s	— —	— —
5	x_t y_t	-0.0030325 -0.0032077	-0.91736 -0.97033	x_s y_s	0.0057185 0.0067668	0.58763 0.69537
6	x_t y_t	-0.0011139 0.00085512	-0.3212 0.24659	x_s y_s	0.0061635 -0.0056457	0.63336 -0.58015
7	x_t y_t	0.0043048 0.0017142	1.2096 0.48168	x_s y_s	-0.013791 0.001748	-1.4172 0.17963
8	x_t y_t	-0.002273 0.00076394	-0.64022 0.21517	x_s y_s	0.010617 -0.0075976	1.091 -0.78073
9	x_t y_t	0.0034443 0.00028881	0.97631 0.081865	x_s y_s	-0.013198 0.0057284	-1.3562 0.58866
10	x_t y_t	-0.0038405 -0.0033504	-1.0817 -0.94369	x_s y_s	0.0086644 0.0057231	0.89036 0.58811

Table 7: Residuals of observations and random vector ***a*** and corresponding *w*-test statistics (Scheme 5) (unit: m)

Point No.	Target system			Start system		
	Coordinate	$\boldsymbol{V}_{Lj}$	$\bar{w}_{Lj}$	Coordinate	$\boldsymbol{V}_{ai}$	$\bar{w}_{ai}$
1	x_t y_t	0.0045553 0.0018727	0.82093 0.33749	x_s y_s	-0.014464 0.0016235	-0.88826 0.09968
2	x_t y_t	**-0.016465** -0.0067103	**-2.9676** -1.2095	x_s y_s	**0.052397** -0.0061015	**3.2177** -0.37462
3	x_t y_t	0.0059838 -0.0014032	1.0783 -0.25286	x_s y_s	-0.026717 0.017593	-1.6407 1.0802
4	x_t y_t	0.0098111 **0.012636**	1.7687 **2.2779**	x_s y_s	-0.013972 **-0.03093**	-0.85803 **-1.8991**
5	x_t y_t	-0.0039266 -0.0067598	-0.70769 -1.2183	x_s y_s	0.0021913 0.019193	0.13456 1.1784
6	x_t y_t	-0.0040698 -0.002357	-0.73355 -0.42483	x_s y_s	0.011557 0.0012864	0.70971 0.078982
7	x_t y_t	0.0012096 -5.41e-005	0.21801 -0.0097511	x_s y_s	-0.004942 0.0026377	-0.30348 0.16195
8	x_t y_t	-0.0035487 0.00097413	-0.63958 0.17556	x_s y_s	0.016128 -0.011002	0.9904 -0.67549
9	x_t y_t	0.0052762 0.0022808	0.9509 0.41105	x_s y_s	-0.016531 0.0014335	-1.0151 0.088012
10	x_t y_t	0.0011737 -0.00047889	0.21154 -0.086313	x_s y_s	-0.0056471 0.0042659	-0.34679 0.26192

Scheme 4: The outlier of magnitude 0.1m is added into the y component of point 4 in both start system and target system.

The concrete results are presented in Table 5. It is not difficult to know $\left|\overline{w}_{a28}\right| = 2.4858 > 1.96$ and $\left|\overline{w}_{L8}\right| = 2.2401 > 1.96$ from Table 5, but the absolute values of other w-statistics are smaller than 1.96. It means that only y component of point 4 contains an outlier, which is consistent with the set simulated case. If we will delete point 4 in both coordinate systems, the new results about the residuals and w-test statistics are obtained, which is displayed in Table 6. It is shown that all $\left|\overline{w}_{ai}\right|$ and $\left|\overline{w}_{Lj}\right|$ are smaller than the threshold value 1.96, which demonstrates that the remaining observations are clean without the effects of outliers.

We just discuss the case that the outlier locates in the same point in two different systems for scheme 1 to 4. In fact, there may be multiple outliers in the different points for the two-dimensional coordinate transformation. Hence, the following scheme 5 is used to assess the efficiency of the proposed procedure for detecting multiple outliers in the partial EIV model.

Scheme 5: In this simulation, two outliers of magnitude 0.1 m are added to the x_s component of point 2 in the start system and the y_t component of point 4 in the target system, respectively.

The detail results about the residuals and w-test statistics are listed in Table 7. $\left|\overline{w}_{a23}\right| = 3.2177 > 1.96$ and $\left|\overline{w}_{L3}\right| = 2.9676 > 1.96$ indicate that the x component of point 2 contains an outlier. On the other hand, due to $\left|\overline{w}_{a28}\right| = 1.8991$ and $\left|\overline{w}_{L8}\right| = 2.2779 > 1.96$, the $y$component of point 4 is probable to be contaminated with an outlier. Because the outlier may locate in the different locations, we will delete point 2 in both coordinate systems firstly. After that, the new results and w-test statistics are obtained, which can be found in Table 8. Apparently, there is an outlier in y component in the start system or target system or both based on the criterion for identifying outlier. As a result, point 4 in both two coordinate systems should be deleted. After removing the assigned outlying observations, the new results about the residuals and w-test statistics are presented in Table 9, which indicates that there is no outlier in the observations of both coordinate systems.

Table 8: Residuals of observations and random vector ***a*** and corresponding *w*-test statistics with deleting the point 2 in both of start system and target system (Scheme 5) (unit: m)

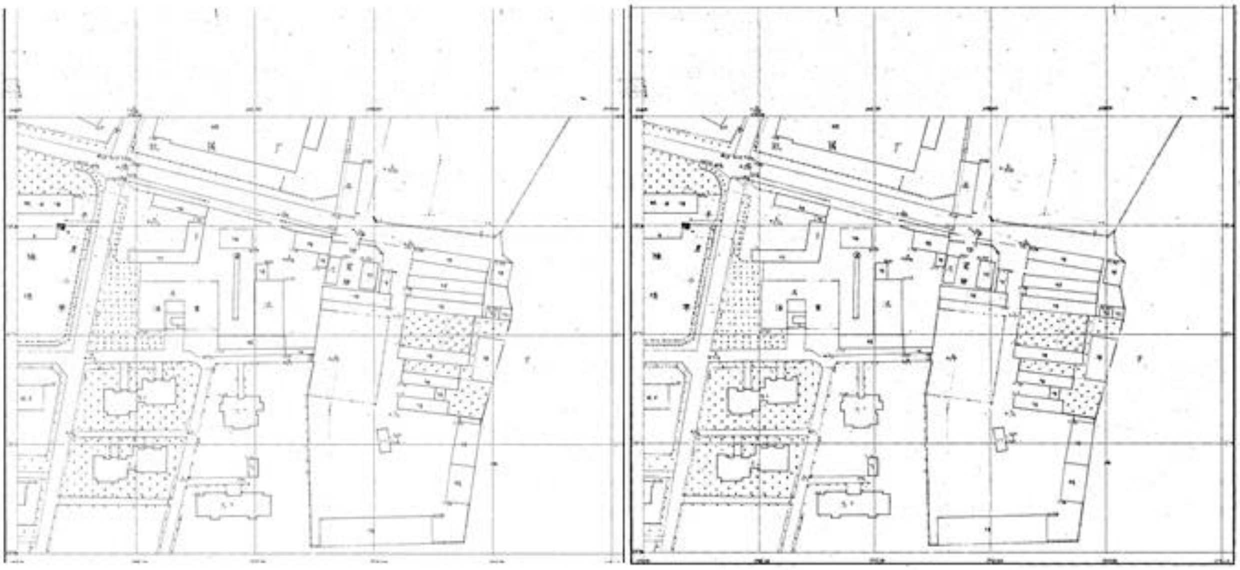
Point No.	Target system			Start system		
	Coordinate	$\boldsymbol{V}_{Lj}$	$\bar{w}_{Lj}$	Coordinate	$\boldsymbol{V}_{ai}$	$\bar{w}_{ai}$
1	x_t	-0.0015997	-0.48961	x_s	0.0051277	0.51616
	y_t	-0.00063521	-0.19442	y_s	-0.00065729	-0.06616
2	x_t	—	—	x_s	—	—
	y_t	—	—	y_s	—	—
3	x_t	-0.00017461	-0.053438	x_s	-0.0071247	-0.71718
	y_t	-0.003914	-1.1979	y_s	0.015311	1.5411
4	x_t	0.005999	1.7512	x_s	-0.0018417	-0.18538
	y_t	**0.011082**	**3.2349**	y_s	**-0.032343**	**-3.2555**
5	x_t	-0.0048223	-1.3711	x_s	0.005046	0.50794
	y_t	-0.0071243	-2.0256	y_s	0.018861	1.8985
6	x_t	-0.0026122	-0.74458	x_s	0.0069248	0.69706
	y_t	-0.0017621	-0.50227	y_s	0.0018267	0.18387
7	x_t	0.0035602	1.0213	x_s	-0.012434	-1.2516
	y_t	0.00090246	0.25889	y_s	0.0035085	0.35315
8	x_t	-0.0020919	-0.59622	x_s	0.011502	1.1578
	y_t	0.0015692	0.44725	y_s	-0.010462	-1.053
9	x_t	0.0043731	1.2433	x_s	-0.013669	-1.3759
	y_t	0.0019111	0.54335	y_s	0.0010985	0.11057
10	x_t	-0.0026315	-0.76792	x_s	0.0064686	0.65114
	y_t	-0.002029	-0.59209	y_s	0.0028559	0.28746

Table 9: Residuals of observations and random vector ***a*** and corresponding *w*-test statistics with deleting point 2 and point 4 in both of start system and target system (Scheme 5) (unit: m)

Point No.	Target system			Start system		
	Coordinate	$\boldsymbol{V}_{Lj}$	$\bar{w}_{Lj}$	Coordinate	$\boldsymbol{V}_{ai}$	$\bar{w}_{ai}$
1	x_t	-0.00070383	-0.23467	x_s	0.004853	0.58449
	y_t	0.00102	0.34007	y_s	-0.0054855	-0.66066
2	x_t	—	—	x_s	—	—
	y_t	—	—	y_s	—	—
3	x_t	0.0027479	1.0525	x_s	-0.0080217	-0.96613
	y_t	0.0014848	0.56868	y_s	-0.00044471	-0.053561
4	x_t	—	—	x_s	—	—
	y_t	—	—	y_s	—	—
5	x_t	-0.002371	-0.79065	x_s	0.0042931	0.51706
	y_t	-0.0025964	-0.8658	y_s	0.0056432	0.67966
6	x_t	-0.0012774	-0.40134	x_s	0.0065153	0.7847
	y_t	0.00070397	0.22118	y_s	-0.0053684	-0.64656
7	x_t	0.0036943	1.1405	x_s	-0.012475	-1.5025
	y_t	0.0011501	0.35506	y_s	0.0027853	0.33546
8	x_t	-0.0027834	-0.85898	x_s	0.011715	1.4109
	y_t	0.00029229	0.090205	y_s	-0.0067323	-0.81083
9	x_t	0.0035459	1.0946	x_s	-0.013415	-1.6157
	y_t	0.00038276	0.11816	y_s	0.005557	0.66929
10	x_t	-0.0028525	-0.8962	x_s	0.0065361	0.7872
	y_t	-0.0024375	-0.76582	y_s	0.0040453	0.48721

4.2 Real data about map rectification

The example is about the map rectification. The 2D affine transformation is used to rectify the map. The scale of map is 1:500 for figure 1.



(a) Distorted map (b) Rectified map

Figure 1: The distorted map and its rectified map using affine transformation

There are ten common points whose theoretical coordinates are previously known, and then we sample their coordinates on the distorted map. The affine transformation is used to rectify the map. The sampled coordinates and theoretical coordinates of common points are treated as the coordinates in the start system and target system, respectively, which is displayed in Table 10. The transformation parameters can be estimated by using the common points with the 2D affine transformation. By employing the proposed algorithm, the residuals and *w*-test statistics of the observations and random vector ***a*** are derived, which is shown in Table 11. Because the *w*-test statistics satisfy $|\overline{w}_{L14}| = 21.838 > 1.96$ and $|\overline{w}_{a34}| = 21.172 > 1.96$, the point 7 is suspected as an outlier and should be deleted. Then the new residuals and *w*-test statistics are obtained, which can be found in Table 12. Due to $|\overline{w}_{a35}| = 1.7622 < 1.96$ for point 9 in the target system, there are no outliers in the observations even if $|\overline{w}_{L15}| = 2.297 > 1.96$ according to the criterion for identifying the outliers in section 3. Therefore, the only outlier is identified. After that, the transformation parameters are estimated by the WTLS method. The results are presented in Table 13. By checking the reliability of the proposed method, the fifteen non-common points are employed to evaluate the performance of the proposed algorithm and RMSE (Root mean square error) is used to judge the influence of outlier for the coordinates. The RMSE for the data-snooping procedure is 0.00892, but is 0.032786 for the WTLS method with outliers. The reason is that the transformation parameters estimated by the WTLS method are disturbed with the outliers.

Table 10: Coordinates of ten common points and fifteen non-common points in both coordinate systems (unit: cm)

Common point				Non-common point			
Start system		Target system		Start system		Target system	
u_s	v_s	u_t	v_t	u_s	v_s	u_t	v_t
77.58677125	87.246990015	34.0	85.0	28.17098162	87.272316176	19.0	85.0
28.13210239	103.72201572	19.0	90.0	44.65580551	87.245720424	24.0	85.0
77.58934311	103.71908529	34.0	90.0	61.10492273	87.244663745	29.0	85.0
28.12765661	120.18027351	19.0	95.0	94.04265529	87.236868084	39.0	85.0
77.606496	120.160256564	34.0	95.0	110.52943807	87.236555391	44.0	85.0
77.61204959	136.623478492	34.0	100	44.62795576	103.714775203	24.0	90.0
28.10320572	154.068706679	19.0	105	61.0884887	103.713279354	29.0	90.0
77.62038088	153.103856739	34.0	105	94.05693187	103.715676958	39.0	90.0
28.08255946	169.529298616	19.0	110.0	110.52749417	103.705978224	44.0	90.0
77.59748129	169.545714888	34.0	110.0	44.64684942	120.159237522	24.0	95.0
				61.11220165	120.128950724	29.0	95.0
				94.0829795	120.1725203	39.0	95.0
				110.5677384	120.165699479	44.0	95.0
				28.11643765	136.617556801	19.0	100.0
				44.60626576	136.611885876	24.0	100.0

Table 11: Residuals of observation and random vector ***a*** and corresponding *w*-test statistics for mapping rectification (unit: cm)

Point No.	Target system			Start system		
	Coordinate	$\boldsymbol{V}_{Lj}$	$\overline{w}_{Lj}$	Coordinate	$\boldsymbol{V}_{ai}$	$\overline{w}_{ai}$
1	x_t	0.0054641	0.66522	x_s	-0.0016239	-0.58441
	y_t	-0.02307	-2.8086	y_s	0.006993	2.5164
2	x_t	-0.0048054	-0.58676	x_s	0.0013928	0.50126
	y_t	0.045573	5.5647	y_s	-0.013814	-4.971
3	x_t	0.0042668	0.4667	x_s	-0.0012697	-0.45693
	y_t	-0.016875	-1.8457	y_s	0.0051151	1.8406
4	x_t	-0.0040557	-0.45558	x_s	0.0011516	0.41444
	y_t	0.055618	6.2476	y_s	-0.016859	-6.0667
5	x_t	-0.00098049	-0.10244	x_s	0.00030013	0.10801
	y_t	-0.0021115	-0.22059	y_s	0.00064008	0.23033
6	x_t	-0.0030061	-0.31407	x_s	0.000902	0.32462
	y_t	0.0065435	0.68363	y_s	-0.0019835	-0.71373
7	x_t	0.0018414	0.20717	x_s	-0.00028704	-0.1033
	y_t	**-0.1941**	**-21.838**	y_s	**0.058838**	**21.172**
8	x_t	-0.0058015	-0.63472	x_s	0.0017438	0.62757
	y_t	0.010432	1.1414	y_s	-0.0031622	-1.1379
9	x_t	0.0070075	0.85308	x_s	-0.0022537	-0.81107
	y_t	0.092938	11.314	y_s	-0.028173	-10.138
10	x_t	6.9481e-005	0.0084591	x_s	-5.6045e-005	-0.02017
	y_t	0.025052	3.0499	y_s	-0.007594	-2.7326

Table 12: Residuals of observation and random vector ***a*** and corresponding *w*-test statistics for mapping rectification with deleting point 7 (unit: cm)

Point No.	Target system			Start system		
	Coordinate	$\boldsymbol{V}_{Lj}$	$\bar{w}_{Lj}$	Coordinate	$\boldsymbol{V}_{ai}$	$\bar{w}_{ai}$
1	x_t	0.005221	1.3414	x_s	-0.0015825	-1.1719
	y_t	0.0026575	0.68277	y_s	-0.00080753	-0.59671
2	x_t	-0.0043552	-1.1521	x_s	0.0013201	0.97753
	y_t	-0.0019646	-0.51971	y_s	0.00059699	0.44113
3	x_t	0.0041199	0.94397	x_s	-0.0012487	0.97753
	y_t	-0.0014459	-0.33129	y_s	0.00043918	0.44113
4	x_t	-0.0035079	-0.86199	x_s	0.0010633	0.78736
	y_t	-0.0022192	-0.54533	y_s	0.00067433	0.49828
5	x_t	-0.0010283	-0.22444	x_s	0.00031166	0.23079
	y_t	0.0030519	0.6661	y_s	-0.00092718	-0.68512
6	x_t	-0.002957	-0.64539	x_s	0.00089626	0.66369
	y_t	0.0014165	0.30916	y_s	-0.00043027	-0.31794
7	x_t	—	—	x_s	—	—
	y_t	—	—	y_s	—	—
8	x_t	-0.0056571	-1.2965	x_s	0.0017147	1.2698
	y_t	-0.0049908	-1.1438	y_s	0.0015164	1.1205
9	x_t	0.0078511	2.297	x_s	-0.0023797	-1.7622
	y_t	0.0041787	1.2226	y_s	-0.0012697	-0.93825
10	x_t	0.00031337	0.080512	x_s	-9.4978e-005	-0.070332
	y_t	-0.00068396	-0.17573	y_s	0.00020779	0.15354

Table 13: Transformation parameters estimated by the WTLS method before deleting outlier and after deleting outlier (unit: cm)

Before deleting outlier	After deleting outlier
0.30309255593699	0.30310519134397
0.00003187394065	0.00002566590120
10.4752902610926	10.47510689386349
0.00139656637130	0.00000654387860
0.30313281644081	0.30381576309241
58.46940628440629	58.48957855017623

5. CONCLUSIONS

The WTLS estimate of the partial EIV model may strongly be influenced by the outliers. The aim of this paper is to develop an approach to detect the outliers in the partial EIV model. Firstly, we propose a two-step iterated method of computing the WTLS estimates for the partial EIV model based on the standard LS theory. Then the corresponding *w*-test statistics are constructed to detect the outliers while the observations, coefficient matrix or both are contaminated with the outliers. If the variance factor is unknown, it may be estimated by the LMS method. Making using of the proposed two-step iterated method, the implement algorithm for detecting the outliers in the partial EIV model is proposed. Through the numerical results with the two-dimensional affine transformation, the identification of outliers is implemented only once through the proposed procedure compared with previously approach while single outlier is considered. For multiple outliers, the repeated test with step by step is suggested. However, we still can't discriminate that the outliers locate in the observation or coefficient matrix or both, which is a very open problem to be discussed in the future.

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